Statistical properties of mesoscopic fluctuations and similarity theory

B. L. Al'tshuler, V. E. Kravtsov, and I. V. Lerner
B. P. Konstantinov Institute of Nuclear Physics, Academy of Sciences of the USSR; Institute of Spectroscopy, Academy of Sciences of the USSR

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The distribution function of mesoscopic fluctuations of the conductance and state density is derived through a renormalization-group analysis in the nonlinear σ model. A hypothesis of an ergodic nature is proved. A deviation from single-parameter scaling is demonstrated.

1. A "mesoscopic" conductor is a disordered conductor which, although large in comparison with the electron mean free path l, is not large enough that we can ignore the difference between its characteristics and their mean values over realizations of the random potential.[1–3] It has been shown[4,5] that the mean square value of the mesoscopic fluctuations, $\delta G = G - \langle G \rangle$, of the total residual conductivity (the conductance) of any sample is given in order of magnitude by

$$\langle \delta G^2 \rangle \sim (e^2/\hbar)^2 , \tag{1}$$

where $\langle \ldots \rangle$ means an average over the realizations of the random potential. This result was derived in first order in a perturbation theory in $g^{-1} \equiv e^2(\pi^2\hbar\langle G \rangle)^{-1} \ll 1$.

We know that when we calculate $\langle G \rangle$ in a two-dimensional system, corrections $\sim g^{-1} \ln(L/l)$, which arise in the higher orders of the perturbation theory, become important as the size (L) of the sample increases. It is important to learn how the summation of these corrections changes the value of $\langle \delta G^2 \rangle$. In particular, we would like to know whether the mesoscopic case can be described by single-parameter scaling.[6]

Experimentally,[1] mesoscopic fluctuations should be manifested as reproducible aperiodic oscillations of G as a function of parameters such as the magnetic field H and the Fermi energy ϵ_F (Refs. 3, 5, 7). Lee and Stone[5] have advanced the hypothesis of an ergodic nature: That taking an average over H or over ϵ_F. In order to test this hypothesis, it is necessary to analyze the higher-order correlation moments. This analysis will also cast light on the nature of the fluctuation distribution function. In the preset letter we analyze the higher orders of the perturbation theory, and we find answers to all these questions.

2. A regular method for constructing a perturbation theory is to take an approach based on the nonlinear σ model (see Ref. 8 and the bibliography there). Calculations which will be published separately lead to the following expression for the correlation function $K_{n,m} = \langle \nu^n G^m \rangle$ of the state density ν and the conductance G in a cube of dimensionality d ($\hbar = 1$):

$$K_{n,m} = \left(\frac{\nu}{N}\right)^n \left(\frac{e}{4\pi N}\right)^{2m} \times \prod_{k \leqslant n < j \leqslant n+m} \left(\frac{\partial}{\partial \omega_k}\right)\left(\mathrm{Sp}\frac{\partial^2}{\partial \mathbf{h}_j^2}\right) \int \frac{\mathscr{D}Q}{Z}\, e^{-F[\omega;\mathbf{h}]} \Bigg|_{(N,\omega,\mathbf{h})=0} \tag{2}$$

Here $Z = \int \mathscr{D}Q \exp(-F[0;0])$, and the generating functional $F[\omega;\mathbf{h}]$ is given in the limit $L/l \to \infty$ by

$$F[\omega;\mathbf{h}] = \int \mathrm{Sp}\left\{\frac{\pi \nu D}{8} (\nabla Q)^2 - \frac{1}{4L^d}\omega \Lambda Q + \frac{g}{8L^d}[\mathbf{h}, Q]^2\right\} d^d r, \tag{3}$$

where D is the diffusion coefficient. The field $Q(\mathbf{r})$ has the structure

$$Q = \tau_\mu Q^\mu_{\alpha\beta;ab;kj}; \quad (Q^\mu)^* = Q^\mu; \quad \mu = 0,1,2,3 \tag{4}$$

$$Q^2 = 1; \quad Q = Q^+; \quad \mathrm{Sp}\, Q = 0. \tag{5}$$

The quantities τ_μ in (4) are quaternions: $\tau_0 = 1; \tau_{1,2,3} = i\sigma_{x,y,z}$ (the σ's are the Pauli matrices). The "replica" indices α and β take on values from 1 to N (in the final results, we have $N = 0$). The indices a and b, each of which takes on two values, stem from the presence of advanced and retarded Green's functions of the electron in the original expressions. We have been forced to introduce some indices (k and j) beyond those in Ref. 8 in order to distinguish between Green's functions referring to different values of ν and G in the correlation function $K_{n,m}$. The sources $\omega\Lambda$ and $\mathbf{h}$, which are independent of $\mathbf{r}$, also have the structure in (4). Here we have $(\omega\Lambda) = \omega_k \tau_0 \delta_{\alpha\beta} (\sigma_z)_{ab} \delta_{kj}$; $\mathbf{h} = -\mathbf{h}^+$, $\mathbf{h}^1 = \mathbf{h}^2 = 0$, and $\mathbf{h}^0$ and $\mathbf{h}^3$ are matrices which are diagonal in terms of the indices k, j and antidiagonal in terms of a, b [the vector indices $\mathbf{h}$ have been omitted from (2)].

In the calculation of $K_{n,m}$, integrals of the type J_4, where $J_s = \int d^d q/q^s$, arise in first order; they lead, in particular, to the result in (1), and they also lead to a corresponding result for[9] $\langle \nu^2 \rangle$. When the next orders are taken into account, $K_{n,m}$ is expressed in terms of the sum over p of terms of the type $(J_4)^{n+m-1}(J_2)^p$, and the contributions containing J_s with $s \geqslant 6$ cancel out. The summation of series in J_2 is known to be equivalent to a renormalization of charges in functional (3).

3. Although functional (3) contains three vertices, in the case $N=0$ it depends on only the renormalization group of the charge g (a dimensionless conductance). The ratio of the charges at the first and third vertices does not change under renormalization-group transformations by virtue of the Einstein relation $g \propto \nu D L^{d-2}$. The coefficient of $\omega\Lambda Q$ also escapes renormalization because of conservation of the number of particles.[8] Consequently, in the expressions derived for $K_{n,m}$ in the first nonvanishing order of the perturbation theory, it is necessary to replace the nucleating charge g_0 by the renormalized charge g [$g = g_0 - \ln(L/l)$ for $d=2$].

Since the charge does not appear in expression (1) for $K_{0,2}$ (as in $K_{1,0}$), that expression remains valid in all orders of the perturbation theory. The other correlation functions depend strongly on g, i.e., on L.

4. At $g \gg 1$, the statistics of the fluctuations of G and ν is approximately Gaussian. Specifically, it can be shown that the cumulant $K^c_{n,m}$, which is found from (2) and (3) by considering only coupled diagrams, is small if $n+m>2$:

$$K^c_{n,m}(K^c_{0,2})^{-m/2}(K^c_{2,0})^{-n/2} \sim g^{-(n+m-2)} \ll 1. \tag{6}$$

We see from (6) that in the case $g \sim 1$, i.e., near an Anderson transition, the distribution function of the mesoscopic fluctuations is highly non-Gaussian.

5. The asymptotic behavior of the distribution function is non-Gaussian at arbitrary values of g. In the calculation of $K^c_{n,m}$ for $n+m \gtrsim g_0$ it is necessary to take into account vertices in $F[\omega;\mathbf{h}]$ in addition to those in (3). These other vertices are proportional to high powers of ω and $\mathbf{h}$, since the charges for these vertices, although small, at the nucleating level, $\sim (l/L)^{2s}$, increase rapidly under renormalization-group transformations. For example, the charge $\Gamma_{0,s}$ at the vertex $\mathrm{Sp}(\mathbf{h},Q]^{2s}$ and thus $K^c_{0,s}$ satisfy the proportionality

$$K^c_{0,s} \propto \Gamma_{0,s} \propto \left(\frac{l}{L}\right)^{2s} \left(\frac{g_0}{g}\right)^{2s^2} \underset{g \gg 1}{\rightarrow} \left(\frac{l}{L}\right)^{2s(1-sg_0^{-1})} \tag{7}$$

An analogous growth law, first established in Ref. 10, holds for the charges for the vertices $\mathrm{Sp}(\omega\Lambda Q)^s$ We see from the explicit functional dependence on g_0 in (7) that single-parameter scaling does not hold.

From (7) we can find the asymptotic behavior of the distribution function $f(\hbar\delta G/e^2)$ at comparatively large values of δG:

$$f(x) \propto \exp\{-A\ln^2(xL^2/l^2)\}; \quad A^{-1} = 8\ln(g_0/g) \to 8g_0^{-1}\ln(L/l). \tag{8}$$

The asymptotic behavior of the distribution function of the fluctuations in the state density, $f(\delta vg/v)$, is also of the form in (8). Distribution (8) holds for $\delta G \gtrsim (Ge^2/\hbar)^{1/2}$. For $d=1$ in the region $g \gg 1$, distribution (8) is the same as the exact distribution found previously.[11]

6. According to the ergodic hypothesis we have $\overline{G^n} = \langle G^n \rangle$, where the superior bar means an average over ϵ_F. To test this hypothesis, it is sufficient to show that the quantity $\{[\langle G^n\rangle - \overline{G^n}]^2\}$, given by

$$\frac{1}{(2E)^2}\int_{\epsilon-E}^{\epsilon+E} d\epsilon_1 d\epsilon_2 \{\langle G^n(\epsilon_1)G^n(\epsilon_2)\rangle - \langle G^n(\epsilon_1)\rangle\langle G^n(\epsilon_2)\rangle\}, \tag{9}$$

tends toward zero with increasing E. In order to analyze correlation functions of the type $\langle \Pi G(\epsilon_j)\rangle$, we should add to (3) a vertex proportional to $\mathrm{Sp}MQ$, where M is a matrix with the structure in (4): $M = \tau_0\delta_{\alpha\beta}\delta_{ab}\delta_{kj}\epsilon_j$. Calculations from (2) and (3) using this vertex show that the integrand in (9) falls off at least as $[|\epsilon_1 - \epsilon_2|L^2/D]^{d-4}$, proving the ergodic nature. The ergodic nature can be proved with respect to an averaging over the magnetic field H in an analogous way.

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