# Long-wave limit in the nonlinear theory of transverse modulations of solitons

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Unstable, long-wave kinks in solitons are analyzed for various wave equations by the method of Whitham (Linear and Nonlinear Waves, Wiley-Interscience, New York, 1974). The problem is reduced to "gas-like" equations which represent new quasi-Chaplygin media {B. A. Trubnikov and S. K. Zhdanov, Fiz. Plazmy **12**, No. 6 (1986); [Sov. J. Plasma Phys. **12**, No. 6 (1986)]; Pis'ma Zh. Eksp. Teor. Fiz. **43**, 178 (1986) [JETP Lett. **43**, 226 (1986)]}.

**1.** The simplified "long-wave" version of the nonlinear evolution of transverse modulations of solitons is interesting primarily because the set of exactly integrable cases[3] is bounded, while the number of "nonintegrable" situations, which are of no less importance, continues to grow.[4–7] In this limit the best method is the method of Whitham,[1] which we will use here. This method makes it possible to go slightly further than is possible in linear perturbation theory.[6,7] Furthermore, a linearization of the equations of the modulations is an alternative and frequently simpler method for studying the linear stage. Since the basic points of this method are generally known,[1] there is no need to discuss them in detail here. We will list several examples which explain the basic situation. We will be concerned primarily with "classical" solitons, described by the Korteweg-de Vries equation, the Kadomtsev-Petviashvili equation, a nonlinear Schrödinger equation, and the sine Gordon equation. A theory of one-dimensional modulations was derived in Refs. 8.

**2.** As a first example we use the isotropic[4,7] Kadomtsev-Petviashili model (with a positive dispersion, $l^2>0$),

$$(u_t+uu_x-c_0l^2u_{xxx})_x=-\frac{c_0}{2}\,\Delta_\perp u;\ \ \Delta_\perp u\equiv u_{yy}+u_{zz}\ , \tag{1}$$

to describe a transverse modulation of a single-parameter (with a fixed wavelength $\lambda$) cnoidal Korteweg-de Vries wave[5] [$K(m)$ and $E(m)$ are elliptic integrals; $m$ is a parameter]:

  

$$u = u_1(x, t) = -A(cn^2(\varphi, m) - \langle cn^2 \rangle); \ \varphi = pl^{-1}(x + ct); \ p = 2K(m)l/\lambda,$$

$$c = 4c_0 p^2(2m-1) - A\langle cn^2 \rangle; \ \langle cn^2 \rangle = (E/K - 1 + m)/m; \ A = 12mc_0 p^2. \tag{2}$$

The Lagrangian corresponding to Eq. (1) is (here and below, $c_0$ is the velocity of sound)

$$\mathscr{L}_1 = u\Phi_t + \frac{1}{3}u^3 + c_0 l^2 u_x^2 + \frac{1}{2}c_0(\nabla_\perp\Phi)^2, \quad u \equiv \Phi_x. \tag{3}$$

Integrating $\mathscr{L}_1$ over the period $\lambda$ with a trial function as in (2), but with an altered phase, $\varphi = pl^{-1}[x + x_0(t, \mathbf{r}_\perp)]$, we find

$$L_1 = \langle \mathscr{L}_1 \rangle \equiv \int_0^\lambda \frac{dx}{\lambda}\mathscr{L}_1 = [x_{0t} + \frac{c_0}{2}(\nabla_\perp x_0)^2]I_1 + c_0 F_1(I_1);$$
$$I_1 = \langle u_1^2 \rangle; \ F_1(I_1) = \langle \frac{u_1^3}{3c_0} + l^2 u_{1x}^2 \rangle. \tag{4}$$

Using (4), and varying the functions $I_1$ and $x_0$, we obtain "gas-like" equations for two-dimensional modulations $[\mathbf{V} = c_0\nabla_\perp x_0, F_1(I_1)$; Fig. 1]:

$$\frac{\partial}{\partial t}I_1 + \mathrm{div}(I_1\mathbf{V}) = 0; \quad \frac{\partial}{\partial t}\mathbf{V} + (\mathbf{V}\nabla)\mathbf{V} = -c_0^2 F_1''(I_1)\nabla I_1. \tag{5}$$

In the soliton limit $m \to 1$, we find $F_1(I_1) \approx -|\mathrm{const}|I_1^{5/3}$ from (2) and (4). This result corresponds to a "monatomic gas" with $\gamma = 5/3$ but with a negative compressibility.[2] In the limit $m \to 0$ $(I_1 \to 0)$, we have $F_1'' \approx -(12c_0^2 l^2 k_0^2)^{-1} = \mathrm{const}, k_0 = 2\pi/\lambda$. This result differs from that derived in Ref. 5, but it is precisely the same as the result of a direct Stokes expansion[1] in the small amplitude.

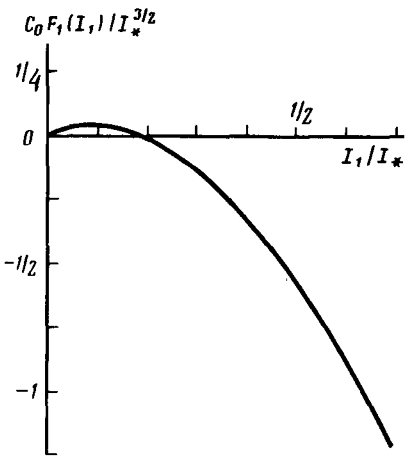


FIG. 1. The function $F_1(I_1)$ for a cnoidal wave, $I_* = (12c_0 l^2 k_0^2)^2, k_0 = 2\pi/\lambda$.

3. We know that a two-dimensional rational Kadomtsev-Petviashvili soliton,[9]

$$u = u_2(x, y, t) = -12c_0 l^2 \frac{\partial^2}{\partial x^2} \ln\left[1 + \left(\frac{x+ct}{\Delta_1}\right)^2 + \left(\frac{y}{\Delta_2}\right)^2\right]; \ \Delta_2^2 = \frac{3l^2c_0^2}{2c^2}; \ \Delta_1^2 = 6^{1/2}\Delta_2 l, \tag{6}$$

is unstable[4] in model (2). Let us describe the growth of kinks of this soliton in the nonlinear stage. Integrating $\mathscr{L}_1$ from (3) with the trial function $u_2[x + x_0(z,t),y,t]$ over the variables $x$ and $y$, we find

$$L_2 = \bar{\bar{\mathscr{L}}}_1 \equiv \int_{-\infty}^{+\infty} dxdy\, \mathscr{L}_1 = [x_{0t} + \frac{c_0}{2} x_{0z}^2] I_2 + c_0 F_2(I_2); \ I_2 = \overline{\overline{u_2^2}}; \tag{7}$$

$$F_2(I_2) = \overline{\overline{u_2^3}}/3c_0 + l^2 \overline{\overline{u_{2x}^2}} + \overline{\overline{\Phi_{2y}^2}}/2 = -3I_2^3/2\pi^2(12c_0 l)^4,$$

which obviously leads us to modulation equations of the type in (5). We introduce the notation $\rho = (c/c_{in})^{1/2} \sim I_2$ ($c_{in}$ is the velocity of the unmodulated wave), and we also introduce $V = c_0 x_{0z}$. We then finally find

$$\rho_t + (\rho V)_z = 0; \ V_t + VV_z = c_{eff}^2 \ \rho\rho_z; \ c_{eff}^2 = 2c_0\, c_{in}, \tag{8}$$

i.e., a one-dimensional "gas" ($\gamma = 3$) with a negative compressibility. In the linear stage we have an aperiodic growth at a rate $\omega^2 = -k_z^2 c_{eff}^2$, which is the same, aside from a change in notation, as that found in Ref. 4.

4. A similar result is found for a transverse modulation of a soliton of the nonlinear Schrödinger equation, which is stable for one-dimensional perturbations.[10] For the nonlinear Schrödinger equation in dimensionless form,

$$i\psi_t + \frac{1}{2}\Delta\psi + \psi|\psi|^2 = 0, \tag{9}$$

the Lagrangian $\mathscr{L}_2$ and the soliton $\psi_0$ are

$$2\mathscr{L}_2 = i(\psi\psi_t^* - \psi^*\psi_t) + \nabla\psi\nabla\psi^* - |\psi|^4; \ \psi_0 = A_0 e^{i\varphi}; \ A_0 = \frac{a}{\cosh ax}; \ \varphi = \frac{a^2}{2} t. \tag{10}$$

Taking an average of $\mathscr{L}_2$ with the trial function $\psi = A_0(x)\exp[i\varphi(t,\mathbf{r}_\perp)]$, we find [cf. (4)]

$$L_3 = \bar{\bar{\mathscr{L}}}_2 \equiv \int_{-\infty}^{+\infty} dx\, \mathscr{L}_2 = [\varphi_t + \frac{1}{2}(\nabla_\perp\varphi)^2] I_3 + F_3(I_3); \ I_3 = \overline{\overline{A_0^2}};$$

$$F_3(I_3) = \frac{1}{2}[\overline{\overline{A_{0x}^2}} - \overline{\overline{A_0^4}}], \tag{11}$$

and modulation equations of the type in (5). Since $I_3 = 2a$, and $F_3 = -a^3/3$, the final form of these equations is[11,12]

$$a_t + \operatorname{div} a\mathbf{V} = 0; \ \mathbf{V}_t + (\mathbf{V}\nabla)\mathbf{V} = a\nabla a; \ \mathbf{V} \equiv \nabla_\perp\varphi. \tag{12}$$

5. A $2\pi$ kink[1] of the sine Gordon equation, $\Delta\varphi - \varphi_{tt} = \sigma \sin\varphi$, with the Lagrangian $-4\sigma\sin^2(\varphi/2)$ is given by

$$\varphi = \varphi_0(x + ct, \Delta) = 4\arctan(\exp\xi); \ \xi = (x+ct)/\Delta; \ c^2 = 1 - \sigma\Delta^2. \tag{13}$$

Taking an average of $\mathscr{L}_3$ with the trial function $\varphi = \varphi_0(x + x_0(t,\mathbf{r}_\perp), \Delta(t,\mathbf{r}_\perp))$; $|\Delta_{t,\mathbf{r}_\perp}| \ll |x_{0t}, \mathbf{r}_\perp|$, we find $\mathscr{L}_3 = \frac{4}{\Delta}[x_{0t}^2 - (\nabla_\perp x_0)^2 - 1 - \sigma\Delta^2]$. We then find an equation of the following type for kink modulations (we are using the notation $\rho = x_{0t}/\Delta$ and $\mathbf{V} = -\nabla_\perp x_0/x_{0t}$:

$$\rho_t + \operatorname{div}\rho\mathbf{V} = 0;\ (\gamma\mathbf{V})_t + \nabla_\perp\gamma = 0;\ \gamma = (1 - V^2 + \sigma\rho^{-2})^{-1/2}. \tag{14}$$

In the limit $V \ll 1, \rho \gg 1$, Eq. (14) reduces to the dynamics of a Chaplygin gas,[2] which is unstable in the case $\sigma = -1$.

**6.** As an example with an anisotropic dispersion, we consider the soliton in (6) of an oblique magnetosonic wave.[5,6] In the notation of Ref. 6, this wave obeys the equation

$$\left[\frac{2}{c_A}\varphi_t + \left[q\varphi^2 - \epsilon\varphi - \frac{c^2}{\omega_{pe}^2}(\sigma^2 - 1)\varphi_{\eta\eta} - \frac{c^2}{\omega_{pi}^2}(2\alpha\varphi_{\eta z} + \varphi_{zz})\right]_\eta\right]_\eta = -\Delta_{x,z}\varphi, \tag{15}$$

where $\alpha$ is the slope, $\sigma^2 = (\alpha\omega_{pe}/\omega_{pi})^2$, and the rest of the notation is explained in Ref. 6. After some obvious changes in notation, we can put (15) in a form close to that of (2):

$$[u_t + uu_x - c_0 l^2 u_{xxx} - c_0 l_1^2(2\alpha u_{zxx} + u_{xzz})]_x = -\frac{1}{2}c_0\Delta_\perp u, \tag{16}$$

where $l_1^2 = c^2/2\omega_{pi}^2$ and $l^2 = (\sigma^2 - 1)c^2/2\omega_{pe}^2 > 0$. Our purpose here is to make (6) a nonsingular solution of (16). The Lagrangian $\mathscr{L}_{\rm an}$ for (16), which generalizes (3), is

$$\mathscr{L}_{\rm an} = \mathscr{L}_1 + \delta\mathscr{L};\quad \delta\mathscr{L} = c_0 l_1^2[u_z^2 - 2\alpha\Phi_z u_{xx}]. \tag{17}$$

In choosing the trial function we take into account the circumstance that the slope of soliton (6) corresponds to the following substitution for Eq. (16) ($\kappa = x_{0z} = \text{const}$):

$$x \to x + \kappa z;\ c \to c + \frac{1}{2}c_0\kappa^2;\ l^2 \to l^2[1 + \kappa(\kappa + 2\alpha)l_1^2/l^4]. \tag{18}$$

Consequently, in contrast with the isotropic case, in which we have $l_1 = 0$, the length $l$ also changes in this case when there is a kink. Taking this circumstance into account, we find $\bar{\mathscr{L}}_{\rm an}$ and then nonlinear modulation equations $[\rho \equiv (c/c_{\rm in})^{1/2},\ V \equiv c_0 x_{0z}]$:

$$(\rho q)_t + [\rho q V + \frac{1}{3}c_{\rm eff}^2\rho^3 q']_z = 0;\ V_t + VV_z = c_{\rm eff}^2\rho\rho_z;$$
$$q = q(V) = 1 + (V^2 + 2\alpha c_0 V)l_1^2/c_0^2 l^2;\ q' \equiv dq/dV. \tag{19}$$

The small-oscillation spectrum for these equations ($V \approx 0$, $\tilde{c} = c - c_{\rm in} \approx 0$) reduces to that found in Ref. 6.

In conclusion we wish to emphasize that the dynamics of the modulations of Korteweg-de Vries, Kadomtsev-Petviashvili, nonlinear-Schrödinger, and sine Gordon solitons conforms to the general theory of quasi-Chaplygin unstable media,[2] as is clear from the discussion above. It thus becomes possible to find several exact solutions, but we do not have space to reproduce them here.

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