Chiral anomaly and the law of conservation of momentum in ^{3}He-A

G. E. Volovik
L. D. Landau Institute of Theoretical and Experimental Physics, Academy of Sciences of the USSR

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The nonconservation of momentum of the fermion vacuum in superfluid ^{3}He-A at $T=0$ is the result of chiral anomaly which is completely analogous to the axial-current anomaly in quantum chromodynamics. The effective "electromagnetic" fields, whose role in ^{3}He-A is played by the order-parameter gradients, produce a momentum source of the form $F^*_{\mu\nu}F^{\mu\nu}$. This Schwinger source describes the transfer of momentum between the fermion vacuum and the excitation system.

At $T=0$ the current in a superfluid ^{3}He-A (the vacuum current) contains, in addition to the usual superfluid current $\rho \mathbf{V}_s$, The orbital currents which are generated because the Cooper pairs have an orbital angular momentum $\hbar$ directed along the common quantization axis $\mathbf{l}(\mathbf{l}^2=1)$:

$$\mathbf{j} = \rho \mathbf{v}_s + \frac{1}{2}\text{curl}\left(\frac{1}{2}\rho\hbar\mathbf{l}\right) - \frac{1}{2}C_0\mathbf{l}\,(\mathbf{l}\cdot\text{curl}\,\mathbf{l}). \tag{1}$$

The last term in the current with C_0, which is approximately equal to the particle density ρ, is an anomalous term in the sense that any attempt to construct a closed phenomenological system of the equations of motion for the variables ρ, $\mathbf{v}_s$, and $\mathbf{l}$ at $T=0$ runs up against this term which causes a breakdown of the law of conservation of the momentum[1] $\mathbf{j}$. Specifically, a source $\mathbf{I}$ appears in the equation for the current [Eq. (1)]:

$$\frac{\partial}{\partial t}\mathbf{j} + \nabla_i \vec{\pi}_i = \mathbf{I}, \tag{2}$$

which in the basic approximation has the form

$$\mathbf{I} = -\frac{1}{2}\mathbf{l}(\mathbf{l}\cdot\text{curl}\,\mathbf{l})\frac{\partial}{\partial t}C_0 - \frac{3}{2}C_0\mathbf{l}\left(\text{curl}\,\mathbf{l}\cdot\frac{\partial}{\partial t}\mathbf{l}\right) + \ldots \tag{3}$$

We must therefore either assume that $C_0=0$, i.e., the microscopic theories leading to $C^0\cong\rho$ are invalid, or admit that even at $T=0$ there are above-vacuum excitations in ^{3}He-A which generate a normal component that absorbs the momentum deficit. In other words, the following equation must hold for the quasiparticle momentum $\mathbf{P}$ if the total current $\mathbf{j}+\mathbf{P}$ is to be conserved:

$$\frac{\partial}{\partial t}\mathbf{P} + \nabla_i \tilde{\vec{\pi}}_i = -\mathbf{I}. \tag{4}$$

As it turned out, the second possibility is realized. On the one hand, it was shown in Refs. 2 and 3 that the existence of an anomalous current with $C_0\cong\rho$ is a conse-

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quence of a chiral anomaly analogous to that which occurs in quantum electrodynamics. This analogy stems from the fact that the Fermi particles in ^{3}He-A are described by the Dirac equation and the role of the gauge field in ^{3}He-A is played by the texture of the vector[4] **l**. On the other hand, Eq. (4) with **I** given by (3) was derived by Combescot and Dombre[4] on the basis of the dynamics of the quasiparticles in the low-frequency hydrodynamic limit ($\omega\tau \ll 1$). This low-frequency regime, however, cannot actually be achieved[4] in ^{3}He-A at $T=0$.

We will show that Eq. (4) is valid even in the general case and that it is a modification for ^{3}He-A of the Schwinger equation for the axial current in quantum electrodynamics[5]:

$$\frac{\partial}{\partial t}\rho_5 + \vec{\nabla}\mathbf{J}_5 = \frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B}, \tag{5}$$

where ρ_5 and $\mathbf{J}_5$ are the components of the axial current $\bar{\psi}\gamma^\mu\gamma_5\psi$, and **E** and **B** contained in the term on the right side of the equation, which describes the anomalous source of the axial current, are the electric and magnetic fields, respectively.

In ^{3}He-A the chiral anomaly stems from the existence of two points on the poles of the Fermi surface where the quasiparticle energy

$$E_{\mathbf{k}} = \left(\left(\frac{k^2}{2m} - \epsilon_F\right)^2 + \Delta_0^2\,[\mathbf{k},\mathbf{l}\,]^2/k_F^2\right)^{1/2}, \quad \epsilon_F = \frac{k_F^2}{2m}$$

vanishes: $\mathbf{k} = \pm k_F\mathbf{l}$. At low temperatures T in comparison with the amplitude of the gap Δ_0 the dominant factor in the dynamics and thermodynamics of the liquid are the excitations with the momenta near the poles. The Bogolyubov equation for quasiparticles near the poles reduces to the Weyl anisotropic equations for charged massless fermions[3]:

$$\begin{gathered}\left[\left(i\frac{\partial}{\partial t} - eA_0\right) - c_{ij}\sigma_i\left(\frac{1}{i}\nabla_j - eA_j\right)\right]\xi = 0,\\ \left[\left(i\frac{\partial}{\partial t} - eA_0\right) + c_{ij}\sigma_i\left(\frac{1}{i}\nabla_j - eA_j\right)\right]\eta = 0.\end{gathered} \tag{6}$$

Here the "speed of light" is an anisotropic tensor:

$$c_{ij} = \frac{k_F}{m}\hat{z}_i\,\hat{z}_j + \frac{\Delta_0}{k_F}(\delta_{ij} - \hat{z}_i\,\hat{z}_j),$$

where the z axis is directed along **l**; the charge $e = \mathbf{k}\cdot\mathbf{l}/k_F$ is $+1$ or -1, depending on which pole is closest to the excitation; ξ and η are the Bogolyubov spinors which describe the quasiparticles with $e = -1$ (right-hand "electrons' and with $e = +1$ (left-hand "positrons"), respectively; and $\vec{\sigma}$ are the Pauli matrices; the gauge fields are expressed in terms of **l** and $\mathbf{v}_s$ as follows:

$$\mathbf{A} = k_F\mathbf{l}, \quad A_0 = k_F\mathbf{l}\cdot\mathbf{v}_s, \quad \mathbf{B} = \text{curl}\,\mathbf{A}, \quad \mathbf{E} = -\vec{\nabla}A_0 - \frac{\partial}{\partial t}\mathbf{A}. \tag{7}$$

A formal analogy between Eqs. (6) and the Dirac equation (the anisotropy of the speed of light can be eliminated by stretching along the z axis) makes it possible to use the Schwinger equation [Eq. (5)] for the axial current, which is expressed in terms of ξ and η in the standard manner:

$$\rho_5 = \langle \xi^+ \xi \rangle - \langle \eta^+ \eta \rangle, \qquad J_{5i} = c_{ij}(\langle \xi^+ \sigma_j \xi \rangle + \langle \eta^+ \sigma_j \eta \rangle). \tag{8}$$

Since $\langle \xi^+ \xi \rangle$ is the number of right-hand quasiparticles with the momenta $-k_F\mathbf{l}$ and $\langle \eta^+ \eta \rangle$ is the number of left-hand quasiparticles with the momenta $k_F\mathbf{l}$, the quantity $\mathbf{P} = -\rho_5 k_F \mathbf{l}$ is the density of the excitation momentum. It therefore follows from Eq. (5) that because of the conversion of left-hand particles to top right-hand particles (by "pumping" the Fermi sea through the "openings" in the poles on the Fermi surface), the excitation momentum $\mathbf{P}$ is not conserved: The vacuum momentum $\mathbf{j}$ is converted to the excitation momentum $\mathbf{P}$.

Let us find the source of the momentum, i.e., the velocity at which the momentum is transferred from the vacuum. Multiplying both sides of Eq. (5) by $-k_F\mathbf{l}$, ignoring the higher-order gradients of the order parameter, and taking into account that $\rho = k_F^3/3\pi^2$, we find Eq. (4) in the form

$$\frac{\partial}{\partial t}\mathbf{P} - \nabla_i(J_{5i} k_F \mathbf{l}) = -\mathbf{I}, \tag{9}$$

where

$$\mathbf{I} = \frac{k_F \mathbf{l}}{2\pi^2}(\mathbf{E}\cdot\mathbf{B})$$

$$= -\frac{1}{2}\mathbf{l}(\mathbf{l}\cdot \mathrm{curl}\,\mathbf{l})\frac{\partial}{\partial t}\rho - \frac{3}{2}\rho\mathbf{l}\left(\mathrm{curl}\,\mathbf{l}\cdot\frac{\partial}{\partial t}\mathbf{l}\right) - \frac{1}{2}\mathbf{l}\left(\vec{\nabla}\rho\cdot\left[\mathbf{l},\frac{\partial}{\partial t}\mathbf{l}\right]\right) + \ldots \,. \tag{10}$$

Expression (10) coincides with expression (3) for the vacuum-current source (1) which was found from the phenomenological hydrodynamics equations that describe the motion of the fermion vacuum. This coincidence shows that the phenomenological equations are valid and that there is an anomalous term with $C_0 = \rho$ in the current (1).

Accordingly, the excitation dynamics must be taken into account even at $T=0$ in the construction of closed dynamics equations for ^{3}He-A. At low frequencies ($\omega\tau \ll 1$), i.e., in the hydrodynamics excitation regime, at $T=0$ ^{3}He-A is described by the variable $\mathbf{v}_n$, which is the velocity of the normal component and which complements the vacuum variables ρ, $\mathbf{v}_s$, and $\mathbf{l}$, with $\mathbf{P} = \rho_n(v_n - v_s)$. The density of the normal component $\vec{\rho}_n$ is nonvanishing even at $T=0$ because of the nonzero state density near the poles of the Fermi surface which appears in the texture due to the presence of the "magnetic" field $\mathbf{B}$ (see Refs. 1, 3, and 4):

$$\rho_n^{ij} = \frac{3}{2}\rho l^i l^j |[\mathbf{l}, \mathrm{curl}\,\mathbf{l}]| / \Delta_0 \,. \tag{11}$$

In the high-frequency limit ($\omega\tau \gg 1$), i.e., in the collisionless limit, which apparently exists only in ^{3}He-A at $T=0$, the excitations are described by the kinetic equation which must be written in such a way that it would automatically take into account the conversion of the vacuum momentum **j** into the excitation momentum **P**.

In ^{3}He-A there are also gauge-field photons **A**, which are orbital waves—collective boson modes—in which the vector **l** oscillates. These waves are described by a Lagrangian which is obtained by means of a gradient expansion of the vacuum energy. At $T = 0$ the dominant term in the Lagrangian is (see Ref. 6)

$$\mathcal{L} = \frac{1}{2} K_3 \left([\mathbf{l}, \operatorname{curl} \mathbf{l}]^2 - \frac{m^2}{k_F^2} (\partial_t \mathbf{l})^2 \right) \cong \frac{K_3}{2k_F^2} (\mathbf{H}_\perp^2 - \mathbf{E}_\perp^2 / c_\parallel^2) \tag{12}$$

with a logarithmically diverging parameter

$$k_3/k_F^3 = \frac{1}{12\pi^2} \ln \frac{\Delta_0^2}{\omega^2}.$$

The photon velocity is anisotropic, $c_{ij} \cong c_\parallel \hat{z}_i \hat{z}_j$, and the logarithmic divergence K_3 is of the same nature as the vacuum polarization, which gives rise to a zero-charge situation in quantum electrodynamics: The effective coupling constant for the coupling of the fermions and photons (orbital waves) at low frequencies (or at long range) is

$$\frac{\tilde{e}^2}{\hbar c_\parallel} = \frac{3\pi}{\ln \frac{\Delta_0^2}{\omega^2}} .$$

There are also fundamental differences between the two field theories which are related primarily to the non-Abelian nature of the Bose fields in ^{3}He-A. These differences account for the different structures of the fermion vacuum in these systems. As a result, the vacuum current in ^{3}He-A differs from the vacuum current in quantum electrodynamics, since these currents are determined by deep vacuum levels. Conversion of the current from the vacuum to the above-vacuum excitations, on the other hand, is described by the same Schwinger term $F^*_{\mu\nu} F^{\mu\nu}$, since it occurs on the "surface" of the vacuum (in ^{3}He-A it occurs on the Fermi surface) and is determined solely by the equations for the excitations near the surface which is identical in each field theory.

The effects of the chiral anomaly for ^{3}He-A can be observed experimentally at a fairly low temperature. In particular, a conversion of the vacuum momentum to the excitation momentum, which is described by the Schwinger equations [Eqs. (9) and (10)], can be detected by measuring the excitation momenta which arise in the time-dependent texture of the vector **l**.

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