The Josephson effect in superfluid helium-3 flowing through a narrow aperture

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The Josephson current is calculated during the flow of helium-3 through a narrow aperture near the critical temperature. The critical current is proportional to $(T_c - T)^2$ in the case of a diffuse reflection of particles from the vessel walls and it is proportional to $(T_c - T)$ in the case of a specular reflection.

The development of experimental technology now makes it possible, in principle, to observe a steady-state Josephson effect in superfluid[1)] helium-3, i.e., a dissipation-free flow of helium-3 through a narrow channel which connects two vessels and which establishes a "weak coupling" between them. In contrast with the superconductivity theory, in which the Josephson effect in weak couplings has been thoroughly studied (see, e.g., the review in Ref. 1), virtually no such calculations have so far been carried out in the theory of super-fluidity of helium-3. The first attempt to calculate the Josephson effect in helium-3 was made by Monien and Tewordt,[2] who considered channel dimensions comparable to $\xi(T)$. This very complex situation required the use of numerical methods. In the present letter we will analyze the formulation of the problem which is similar to a short, weak, clean coupling in superconductors, where an analytic expression for the Josephson current can be obtained.

We will consider two vessels, 1 and 2, separated by a partition with a small aperture whose radius and length are small in comparison with ξ_0. This situation is illustrated in Fig. 1, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors of the inner normal to the surface of the partition, as viewed from vessels 1 and 2, respectively. We will restrict our analysis to the temperatures approximately equal to the critical temperature. Because of the small size of the aperture, we can assume that the state of helium-3 in vessel 2

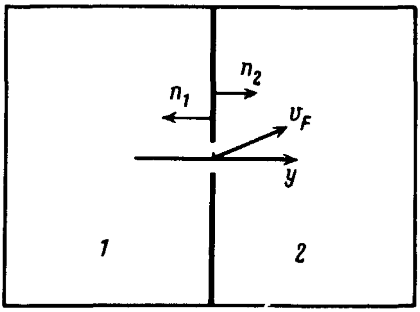

FIG. 1.

does not depend on its state in vessel 1, and vice versa, and that the superfluid velocity v^s near the aperture is low in comparison with the critical decoupling rate v_c. At large spatial separations from the aperture compared with $\xi(T)$, where the density of the superfluid component is $\rho_s = \text{const}$, the incompressibility condition div $j_s = 0$ gives $v^s_\rho = I/2\pi\rho_s r^2$, where v^s_ρ is the velocity in the direction of the aperture, r is the distance from the aperture, and $I = j_0 S$ is the current that flows through the aperture with a cross-sectional area S. We thus find the estimate $v^s_\rho/v_c \sim (j_0/j_c)R^2/r^2$, where j_c is the density of the decoupling current, and R is the aperture radius. The superfluid velocity v^s is low compared with v_c if $R^2/\xi^2(T) \ll j_c/j_0$. It follows from the results obtained below that $R \ll \xi_0$ this inequality is always valid. We can thus ignore the flow of current through the aperture in each vessel and we can assume that the order parameter near the aperture has the same value it would have if the aperture were not there.

The equation for the semiclassical Green's functions in the vessel is[3]

$$-i\mathbf{v}_F\nabla \hat{f}^{R(A)}_\epsilon(\mathbf{p},\mathbf{r}) - 2\epsilon \hat{f}^{R(A)}_\epsilon(\mathbf{p},\mathbf{r}) + \hat{g}^{R(A)}_\epsilon(\mathbf{p},\mathbf{r})\hat{\Delta}_\mathbf{p} - \hat{\Delta}_\mathbf{p}\hat{\bar{g}}^{R(A)}_\epsilon(\mathbf{p},\mathbf{r}) = 0, \qquad (1)$$

where $\hat{\Delta}_\mathbf{p} = \hat{A}\mathbf{p} = i\hat{\sigma}^{(\alpha)}\hat{\sigma}^{(2)}A_{\alpha i}p_i$, and $\mathbf{p}$ is the unit vector in the direction of $\mathbf{v}_F$. Near the critical temperature, Eq. (1) can easily be solved by expanding its variables in powers of $\hat{\Delta}_\mathbf{p}$:

$$\hat{f}^{R(L)} = \pm\left(\frac{\hat{\Delta}_\mathbf{p}}{\epsilon} - \frac{i}{2\epsilon^2}\mathbf{v}_F\nabla\hat{\Delta}_\mathbf{p}\right)\left[1-\exp\left(\pm\frac{2i\epsilon}{v_F}|\mathbf{r}-\mathbf{r}_\pm|\right)\right]$$
$$+\left(\frac{|\mathbf{r}-\mathbf{r}_\pm|}{\epsilon}\mathbf{p}\nabla\hat{\Delta}_\mathbf{p} + C^{R(L)}\right)\exp\left(\pm\frac{2i\epsilon}{v_F}|\mathbf{r}-\mathbf{r}_\pm|\right). \qquad (2)$$

Here the upper sign is used for the retarded Green's function (R) and the lower sign is used for the leading Green's function (L). The constants $C^{R(L)}$, are determined from the boundary conditions for the functions $f^{R(L)}$, with allowance for the scattering by the walls of the vessel (see, e.g., Refs. 4 and 5). In our case, these constants are unimportant. In Eq. (2) $\mathbf{r}_\pm$ are the radii vectors of the points at which the straight line, which goes through the point $\mathbf{r}$ at which the particle is located and which is parallel to the velocity $\mathbf{v}_F$, crosses the walls of the vessel. The radius vector $\mathbf{r}_+$ is situated at the wall where $\mathbf{n}\mathbf{v}_F > 0$ and r_- is situated at the wall where $\mathbf{n}\mathbf{v}_F < 0$ ($\mathbf{n}$ is the vector of the normal to the surface). The equations for $\hat{f}^{+R(L)}$ are derived from (2) through the substitution $\mathbf{v}_F\hat{\nabla}\to -\mathbf{v}_F\hat{\nabla}, \hat{\Delta}\to\hat{\Delta}^*$, and $\mathbf{r}_\pm \to \mathbf{r}_\mp$.

Since there is no reflection from the wall at the aperture, at $v_y > 0$ the value of the function f^R, as can be seen from Fig. 1 and Eq. (2), for example, corresponds to f_1^R $(\mathbf{n}_1\mathbf{v} < 0)$ and at $v_y < 0$ it corresponds to $\hat{f}_2^R(\mathbf{n}_2\mathbf{v} < 0)$. The indices 1 and 2 show that the functions f in (2) are used when $\hat{\Delta} = \hat{\Delta}_1$ and $\hat{\Delta} = \hat{\Delta}_2$ in vessels 1 and 2, respectively. By analogy with the above reasoning, we find the values of the functions f and f^+ in the aperture:

$$\hat{f}^R(v_y > 0) = \frac{\hat{\Delta}_1}{\epsilon} - \frac{i}{2\epsilon^2}\mathbf{v}_F\nabla\Delta_1 ; \quad \hat{f}^R(v_y < 0) = \frac{\hat{\Delta}_2}{\epsilon} - \frac{i}{2\epsilon^2}\mathbf{v}_F\nabla\hat{\Delta}_2 ;$$

$$\hat{f}^{+R}(v_y > 0) = \frac{\hat{\Delta}_2^*}{\epsilon} + \frac{i}{2\epsilon^2}\mathbf{v}_F\nabla\hat{\Delta}_2^* ; \quad \hat{f}^{+R}(v_y < 0) = \frac{\hat{\Delta}_1^*}{\epsilon} + \frac{i}{2\epsilon^2}\mathbf{v}_F\nabla\hat{\Delta}_1^* . \quad (3)$$

The leading functions can be found from (3) by changing the sign and through the substitution $\hat{\Delta}_1 \rightleftarrows \hat{\Delta}_2$. The functions $\hat{g}^{R(L)} = -\hat{g}^{R(L)}$ can be found from the condition $g^2 - ff^+ = 1$. Using a standard equation to calculate the current that flows through the aperture, we find

$$I = -\frac{S\nu(0)mv_F\pi i}{128T}\mathrm{Tr}\left\{(\hat{\mathbf{A}}_1^*\hat{\mathbf{A}}_2 - \mathrm{c.c.})\right.$$

$$\left. + \frac{28\zeta(3)v_F}{15\pi^3 T}\left(\frac{\partial\hat{\mathbf{A}}_1^*}{\partial\rho_1}\mathbf{A}_2 + \mathbf{A}_1^*\frac{\partial\mathbf{A}_2}{\partial\rho_2} - \mathrm{c.c.}\right) + \frac{v_F^2}{144T^2}\left(\frac{\partial\mathbf{A}_1^*}{\partial\rho_1}\frac{\partial\mathbf{A}_2}{\partial\rho_2} - \mathrm{c.c.}\right)\right\}. \quad (4)$$

Here $\partial/\partial\rho_{1,2}$ are the derivatives of the inner normals $n_{1,2}$ with respect to the partition between the vessels.

The values of the order parameter, $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$, and of its derivatives are determined at the surface of the partition near the aperture. The boundary conditions require that $\hat{\mathbf{A}}_1\mathbf{n}_1 = \hat{\mathbf{A}}_2\mathbf{n}_2 = 0$ at the wall.[6] In the case of a diffusion reflection of particles from the surface of the partition, the components $\hat{A}_T$ also turn out to be small and tangential to the wall. According to Ref. 6, at the wall we have $\hat{\mathbf{A}}_T = b_T(\partial\hat{\mathbf{A}}_T/\partial\rho)$, where $b_T \sim \xi_0 = v_F/2\pi T_c$. We thus find $A_T \sim [\xi_0/\xi(T)]\Delta \sim \Delta^2/T$, where $\Delta = (A_{\alpha i}A_{\alpha i}^*)^{1/2}$ is the value of the modulus of the order parameter in the vessel. Calculations show that

$$b_T = \frac{56\zeta(3)}{15\pi^2}\xi_0 \approx 0.45\,\xi_0 .$$

This value is slightly different from the value $b_T = 0.54\xi_0$ obtained in Ref. 6. As a result, we find the following expression from (4) in the case of a diffuse reflection:

$$I = -S\nu(0)m\,\xi_0^3\,\alpha i\,\mathrm{Tr}\left(\frac{\partial\hat{\mathbf{A}}_1^*}{\partial\rho_1}\frac{\partial\hat{\mathbf{A}}_2}{\partial\rho_2} - \frac{\partial\hat{\mathbf{A}}_1}{\partial\rho_1}\frac{\partial\hat{\mathbf{A}}_2^*}{\partial\rho_2}\right). \quad (5)$$

The numerical coefficient is

$$\alpha = \frac{\pi^4}{64.36} + \frac{3[7\zeta(3)]^2}{225\,\pi^2} \simeq 0.14$$

Equation (5) can be written in the form $I = I_0 \sin(\varphi_2 - \varphi_1)$, where the Josephson critical current

$$I_0 \sim S\nu(0)\, m v_F T \left(\frac{\Delta_1 \Delta_2}{T^2}\right)^2$$

is proportional to $(T_c - T)^2$.

Since the tangential component $\hat{A}_T$ near the wall remains finite in the case of specular reflection from the surface of the partition, we must retain only the first term in Eq. (4):

$$I = -\frac{S\nu(0) m v_F \pi i}{64\, T}\,\mathrm{Tr}\,(\hat{\mathbf{A}}_1^* \hat{\mathbf{A}}_2 - \hat{\mathbf{A}}_1 \hat{\mathbf{A}}_2^*). \qquad (6)$$

The Josephson critical current

$$I_0 \sim S\nu(0) m v_F T \left(\frac{\Delta_1 \Delta_2}{T^2}\right)$$

is proportional to $(T_c - T)$. Equation (6) is similar to the result for a short, smooth, weak coupling in the theory of superconductivity.[1] The temperature dependence of the Josephson critical current differs in each case from the dependence of the critical decoupling current, $j_c \propto (T_c - T)^{3/2}$.

In the case of the A phase, the Josephson current [Eqs. (5) and (6)] may vanish in the approximation considered by us if the vectors of the anisotropy, $\mathbf{l}_1$ and $\mathbf{l}_2$, in vessels 1 and 2 are directed opposite to each other near the partition, i.e., $\mathbf{l}_1 = -\mathbf{l}_2$. This situation is consistent with the assertion made by Thuneberg and Kurkijärvi.[7]

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