Rigorously solvable model of spin glass

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We propose a model that lends itself to a rigorous solution for a disordered spin system with an infinite-radius interaction, in which a phase transition into the spin-glass state takes place and in which the heat capacity exhibits the correct behavior at low temperatures.

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Dilute solutions of transition-metal atoms (Fe, Co, Mn) in paramagnetic metals (Cu, Au) have been the object of experimental and theoretical research for a number of years (see[1–3] and the references given there). These systems have a large number of interesting properties, noteworthy among which is the very sharp peak in the plot of the magnetic susceptibility $\chi(T)$ in zero field and the linear dependence of the heat capacity on the temperature as $T \to 0$, with a coefficient that does not depend on the impurity concentration. It has been clear for quite a time that these and many other properties of these solutions are due to the indirect RKKY (Ruderman-Kittel-Kasuya-Yoshida) interaction between the impurity atoms via exchange of the matrix conduction electrons, and is given by

$$J(r_{ij})S_i S_j \;. \qquad J(r) = (k_F r)^{-3}\cos(2k_F r), \tag{1}$$

where S_i are the spins of the impurity atoms and k_F is the Fermi momentum.

The rapidly oscillating and weakly decreasing character of (1), as well as random character of the distribution of the impurities, cause at sufficiently low temperatures "freezing" of their spins in random directions, accompanied thus by an increase of χ. The resultant magnetic structure is called spin glass. It constitutes a "conglomerate" of blocks of spins that are relatively little disoriented, the total orientation of which, however, changes from block to block in such a way that the macroscopic moment of the system turns out to be equal to zero. It was proposed in[1] to regard this "freezing" of the spins as a certain phase transition. But since it is unclear how to solve the statistical-physics problem corresponding to the interaction (1), it was proposed in[1] to replace $J(r_{ij})$ by independent Gaussian random quantities J_{ij} with zero mean value ($\langle J_{ij}\rangle = 0$), and this, in the opinion of the authors of[1], should simulate the rapid oscillations (1). According to[1] this model is subject, in the self-consistent field approximation, to a phase transition accompanied by a break in $\chi(T)$, which the authors interpret as a transition to the spin-glass state. An attempt was made in[2] to impart an asymptotically exact meaning to the results of[1] by introducing in front of J_{ij} a factor $N^{-1/2}$, in analogy with the Curie-Weiss theory (N is the total number of sites). However, the calculations in[2] seem to be incorrect, since, as noted by the authors themselves, they yield a negative entropy as $T \to 0$.

We propose in this paper another model, which is asymptotically exact as $N \to \infty$, in which the random exchange integrals J_{ij} are

$$J_{ij} \cong -N^{-1} \sum_{1}^{n_1} f_k \alpha_i^{(k)} \alpha_j^{(k)} + N^{-1} \sum_{1}^{n_2} a_k \alpha_i^{(k+n_1)} \alpha_j^{(k+n_1)}, \qquad (2)$$

where n_1 and n_2 are fixed numbers that which specify the number of negative-definite (ferromagnetic) and positive-definite (antiferromagnetic) harmonics in the interaction J_{ij}, while f_k and a_k are positive parameters (coupling constants) and $\alpha_i^{(k)}$ are random and generally speaking statistically dependent quantities, the joint distribution of which is invariant to the substitution $\alpha_i^{(k)} \to \alpha_{i+1}^{(k)}$ and are such that the statistical correlations between them vanish as $|i-j| \to \infty$.

By using a method that generalizes the method developed in[4], it can be shown that as $N \to \infty$, in each fixed realization of the quantities $\alpha_i^{(k)}$, the free energy corresponding to the interaction (2) tends to a nonrandom limit

$$f = \min_{\{\mathbf{F}_k\}} \max_{\{\mathbf{A}_k\}} \left\{ \frac{1}{2} \sum_{1}^{n_1} f_k \mathbf{F}_k^2 - \frac{1}{2} \sum_{1}^{n_2} a_k \mathbf{A}_k^2 + \langle \phi(|\vec{\gamma}|) \rangle \right\}, \qquad (3)$$

where

$$\vec{\gamma} = \sum_{1}^{n_1} f_k \mathbf{F}_k \alpha_i^{(k)} - \sum_{1}^{n_2} a_k \mathbf{A}_k \alpha_i^{(k+n_1)} + \mathbf{h},$$

and $\mathbf{F}_k$ and $\mathbf{A}_k$ are D-dimensional vectors (order parameters), $\mathbf{h}$ is the external field, $\phi(|\chi|)$ is the free energy of one spin in the field, equal to $-\beta^{-1}\ln\{2\cosh(\beta\gamma)\}$ in the Ising model ($D = 1$), to $-\beta^{-1}\ln\{[\sinh(\beta\gamma)]/\beta\gamma\}$ in the classical Heisenberg model ($D = 3$), and to $-\beta^{-1}\{\sinh[\beta\gamma(s+\tfrac{1}{2})]/\sinh(\beta\gamma/2)\}$ in the Heisenberg quantum model with spin s, where B is the reciprocal temperature.

We consider some particular cases of (3), assuming that the probability density $p(\alpha)$ of each $\alpha_i^{(k)}$ is given by

TABLE I.

Quantity	T_c	$\chi'(T_c+0)$	$\Delta\chi'(T_c)$	$C(T),\ T \ll T_c$
Classical model, D–dimensionality of spin unit vector	$\frac{J a_2}{D}$	$-\frac{Dc}{J^2 a_2^2}$	$-\frac{3Dc}{J^2 a_4}$	$C(T) \approx c\frac{D-1}{2}$
Quantum model, s–value of spin	$\frac{Js(s+1)a_2}{3}$	$\frac{-3c}{J^2 s(s+1)a_2^2}$	$\frac{-9c}{J^2 s(s+1)a_4}$	$C(T)T^{-1} \approx \frac{2\pi^2 q(0)}{3J^2 \langle\lvert a\rvert\rangle_q (2s+1)}$

$$p(a) = (1-c)\delta(a) + cq(a), \tag{4}$$

where $q(\alpha) \geq 0$, and $\int q(\alpha)\,d\alpha = 1$. This form of $p(\alpha)$ corresponds to the fact that if c is the impurity concentration, then each of them has a probability of c or $1-c$ of being present or absent in any of the lattice sites. By the same token, we have introduced into the theory the dependence on the impurity concentration, a dependence missing from[1,2].

Assume that in (3) we have only $f_1 \equiv J \neq 0$, i.e., $n_1 = 1$ and $n_2 = 0$. Then, if $\langle\alpha\rangle \neq 0$, we arrive at the theory analogous to the molecular-field theory with a critical temperature proportional to the concentration, $T_c = cJ\langle\alpha^2\rangle_q \langle\alpha^e\rangle_q = \int \alpha^e q\,d\alpha$, a spontaneous magnetization that differs from zero at $T < T_c$, and the usual behavior of $\chi(T)$ as $T \to T_c \pm 0$, namely $\chi \approx \langle\alpha\rangle^2[\langle\alpha^2\rangle \mid T - T_c \mid A_\pm]^{-1}$, with $A_+ = A_- \times 2 = 1$ in the Ising model and in the Heisenberg quantum model (cf. [5]). This is a disordered ferromagnet (DF). On the other hand if $q(\alpha)$ in (4) is an even function, then the spontaneous magnetization is equal to zero also at $T < T_c$, which $\chi(T)$ is continuous at the point $T = T_c$ but has kink. This is spin glass. The magnitude of the kink in $\chi(T)$, which depends on the form of $\phi(\gamma)$, as well as some other characteristics of the considered models, are given in Table I, where $\alpha_l \equiv \langle\alpha^l\rangle = c\langle\alpha^l\rangle_q$ and $\Delta\chi' = \chi'(T_c+0) - \chi'(T_c-0)$. It is seen from Table I that at $T \sim T_c$ all the models behave qualitatively in the same manner, and the linear behavior of the heat capacity observed at low temperatures with a slope that is independent of the concentration is present in the Ising model [here $C(T)T^{-1} \approx \pi^2 q(0)/12J^2\langle\lvert\alpha\rvert\rangle_q$ as $T \to 0$] and in the quantum models.

Let us describe briefly also the case[1)] $n = n_1 + n_2 = 2$.

a) $n_1 = 2$, $n_2 = 0$. There are two phase transitions. If the random quantity $\alpha^{(1)}$ is asymmetrically distributed, then in the first transition the paramagnetic state gives way to the disordered-ferromagnet state, while the second transition is of the $DF_1 \to DF_2$ type, i.e., from one phase of the disordered-ferromagnet type to another of the same type. If, however, the $\alpha_i^{(1)}$ are symmetrical, then in the first transition the paramagnetic state gives way to a spin-glass (SG) state, and the second transition is of the SG—DF type if the $\alpha_i^{(2)}$ are symmetrical and of the $SG_1 \to SG_2$ type if the $\alpha_i^{(2)}$ are symmetrical. The behavior of the thermodynamic quantities in the vicinity of both critical temperatures is qualitatively the same as in the case $n_1 = 1$, $n_2 = 0$ considered above.

b) $n_1=n_2=1$. In a zero field, the model behaves in the same manner as at $n_1=1$ and $n_2=0$.

c) $n_1=0$, $n_2=2$. The model coincides with the system of interacting spins.

The results above can be used also to describe disordered systems, in which the role of the spins is played by electric dipole moments (see, e.g., [7] concerning such systems).

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1)As this paper was being readied for press, the authors learned of an article[6] dealing with a particular case of this model, when $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ each assume two specially selected values. The method of[6] is different in principle and does not make it possible to deal with distributions of the type (4).

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