Nonlocal condensates and QCD sum rules for the pion wave function

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The QCD sum rules for the moments of the pion wave function $\varphi_\pi(x)$ are extremely sensitive to the coordinate dependence of "nonlocal condensates": $\langle \bar{q}(0)q(z)\rangle \equiv M(z^2)$, $\langle \bar{q}(0)\gamma_\mu\, q(z)\rangle$, etc. Modified sum rules are derived. An explicit expression $\varphi_\pi(x) \simeq (8/\pi) f_\pi \sqrt{x(1-x)}$ is found for distributions $M(z^2)$ having the width dictated by the standard value ($\lambda^2 = 0.4$ GeV^2) of the ratio $\lambda^2 = \langle \bar{q}D^2q\rangle . \langle \bar{q}q\rangle$.

An important problem in the theory of strong interactions is to calculate the distribution functions $f_{q/H}(x), f_{g/H}(x)$ and the wave functions $\varphi_\pi(x)$, $\varphi_N(x_1,x_2,x_3)$, of the hadrons, which carry information on nonperturbative aspects of quark-gluon dynamics, from the first principles of quantum chromodynamics (QCD). The method of QCD sum rules is promising for calculating the lower moments of these functions.[1] For example, the zeroth moment of $\varphi_\pi(x)$ (i.e., the constant f_π) was derived within 5% of Ref. 1. In Ref. 2, the sum rules for f_π were formally generalized to the succeeding moments of the function $\varphi_\pi(x)$. Information on the nonperturbative dynamics in the method of QCD sum rules accumulates in a power series in the vacuum expectation values of local operators. This series also determines the magnitudes of the hadron characteristics. However, since $\varphi_\pi(x)$ is a function which parametrizes the matrix element of a nonlocal operator,[1)]

$$\langle \bar{u}(0)\gamma_5\gamma_\mu d(z)|P\rangle|_{z^2=0} = iP_\mu \int_0^1 e^{i(Pz)x}\varphi_\pi(x)dx \tag{1}$$

the following question arises: Would it be possible to extract reliable information on very nonlocal entities in the standard version of the method of sum rules, restricted to the simplest local vacuum expectation values $\langle\bar{q}(0)q(0)\rangle$, $\langle G_{\mu\nu}(0)G_{\mu\nu}(0)\rangle$, etc? Or would it be necessary to take into account nonlocal vacuum expectation values, $BC\langle\bar{q}(y)q(z)\rangle$, $\langle G_{\mu\nu}(y)G_{\mu\nu}(z)\rangle$,... (especially since these nonlocal values are the starting points for any calculations by the method of QCD sum rules, and the local vacuum expectation values arise from them after an expansion in a Taylor series)?

To study bilocal vacuum expectation values, it is convenient to introduce the parametrization[2)]

$$\langle\bar{q}(0)q(z)\rangle = \langle\bar{q}q\rangle\int_0^\infty e^{\nu z^2/4}\Phi(\nu)d\nu, \tag{2}$$

which has the structure of the α representation for a propagator. An expansion of $\langle\bar{q}(0)q(z)\rangle$ in local operators corresponds to an expansion of $\Phi(\nu)$ in $\delta^{(n)}(\nu)$:

$$\Phi(\nu) = \delta(\nu) - (\lambda^2/2)\delta'(\nu) + \cdots,$$

where $\lambda^2 \equiv \langle\bar{q}D^2q\rangle/\langle\bar{q}q\rangle$ is the expectation value of the virtuality of the vacuum quarks.

For a bilocal vacuum expectation value containing a γ matrix,

$$\langle\bar{q}(0)\gamma_\mu q(z)\rangle = -iz_\mu\int_0^\infty e^{\nu z^2/4}\Psi(\nu)d\nu \tag{3}$$

the zeroth moment of $\Psi(\nu)$ vanishes in the limit of massless quarks, so that the series in $\delta^{(n)}(\nu)$ for $\Psi(\nu)$ begins with $\delta'(\nu)$:

$$\Psi(\nu) = A\left[\delta'(\nu) - \frac{57}{80}\lambda^2\delta''(\nu)\right],$$

where

$$A = \frac{2}{81}\pi\alpha_s\langle\bar{q}q\rangle^2$$

Contributions proportional to $\alpha_s\langle\bar{q}q\rangle^2$ also arise from "trilocal" condensates

$$\langle\bar{q}(0)\gamma_\mu\binom{\gamma_5}{1}A_\nu(y)q(z)\rangle,$$

which are parametrized by a triple integral representation of the same type.

The role of the functions $\Phi(\nu)$,$\Psi(\nu)$,... is particularly obvious if we write the sum rules directly for the wave function ($\bar{x}\equiv 1-x$,...)

$$f_\pi \varphi_\pi(x) = \frac{3M^2}{4\pi^2} x\bar{x}(1 - e^{-S_0/M^2}) + 4\Psi(xM^2)$$
$$+ \frac{16}{9}\pi\alpha_s\langle\bar{q}q\rangle^2 \int_0^1 \bar{x}\bar{y}dy \int_0^1 da \int_0^1 ab\, \Phi(xM^2/a)\Phi(yM^2/b) \cdot \frac{\theta(x>\bar{y})\theta(a<\bar{b}) + \theta(x<\bar{y})\theta(a>\bar{b})}{|xy\bar{a}\bar{b} - \bar{x}\bar{y}ab|} + \quad (4)$$

$+$ (trilocals) $+$ $O(GG)$ $+$ $(x \to \bar{x})$. The longitudinal-momentum distribution of the quarks of the pion is thus unambiguously related to the virtuality distribution of the vacuum fields.

The standard sum rules[1,2] follow from (4) if for $\Phi(\nu),\Psi(\nu)$ we take the first terms of the expansion in $\delta^{(n)}(\nu)$. Such a model would obviously be too crude when the expectation value of the virtuality (λ^2) of the vacuum quarks (and/or the gluons) is not small in comparison with the characteristic hadronic scale value, $S_0^{(N=0)} \simeq 0.75$ GeV^2. The existing estimates[5] yield $\lambda^2 = 0.4 \pm 0.1$ GeV$^2 \sim S_0^{(N=0)}$. In a situation of this sort, it is necessary to switch from a standard expansion in nonlocal vacuum expectation values to an expansion in which the presence of a high expectation vaue of the virtuality for the vacuum fields is taken into account even in the lowest term. In other words, for the functions $\langle\bar{q}(0)q(z)\rangle \equiv M(z^2)$ with a finite ($\sim 1/\mu$) correlation length for the vacuum fluctuations, it is clearly preferable to expand $\Phi(\nu)$ in a series in $\delta^{(n)}(\nu - \mu^2)$, whose first term incorporates the basic effect, which stems from the finite width of the function $M(z^2)$, while the following terms make it possible to deal with a deviation of $M(z^2)$ from a Gaussian shape. We therefore take $\Phi(\nu)$ in the form $\delta(\nu - \lambda^2/2)$, and $\Psi(\nu)$ in the form $A\delta'(\nu - \frac{57}{80}\lambda^2)$. The magnitudes of the shifts are evidently determined by the second terms of the expansion of $\Phi(\nu),\Psi(\nu)$ in $\delta^{(n)}(\nu)$. In an analogous way, we can construct model δ-functions for trilocal and gluon vacuum expectation values. As a result, we find the following sum rule for the moments of the pion wave function from $\varphi_\pi(x)$:

$$4\pi^2 f_\pi^2 \langle\xi^N\rangle = \frac{3M^2}{(N+1)(N+3)}(1 - e^{-S_0/M^2}) + \frac{\pi\alpha_s}{3M^2}\langle GG\rangle\delta_G^N$$
$$+ \frac{64}{81}\pi^3\alpha_s\langle\bar{q}q\rangle^2\left\{\frac{1}{M^4}\sum_{i=0}^{2}\Delta_i\delta_i^N\left(1 + 2N\frac{\Delta_i}{\delta_i}\right)\theta(\delta_i > 0) + \frac{18}{\lambda^4}\left[\frac{1-\delta_1^{N+1}}{N+1} - \frac{1-\delta_1^{N+2}}{N+2}\right]\right\}, \quad (5)$$

where $\xi = x - \bar{x}$, $\Delta_i = 1 - a_i\lambda^2/2M^2$, $\delta_i = 1 - a_i\lambda^2/M^2$, $a_0 = 57/80$, $a_1 = 1$, and $a_2 = 23/24$.

In the case $\lambda^2 = 0$, expression (5) becomes the Chernyak-Zhitnitsky sum rule,[2] and the term proportional to λ^2 in an expansion of (5) in a series of λ^2 yields the (model-independent) value of the $\alpha_s\langle\bar{q}D^2q\rangle\langle\bar{q}q\rangle$ contribution. It should be noted that the latter completely cancels the $\alpha_s\langle\bar{q}q\rangle^2$ contribution at $M^2 \simeq 0.6$ GeV^2.

In the Chernyak-Zhitnitsky sum rule,[2] the relative contribution of the $\alpha_S\langle\bar{q}q\rangle^2$ and $\alpha_s\langle GG\rangle$ corrections increases rapidly with N. As a result, the scale $S_0^{(N=0)}$ at

$N = 2$ (4) is 2.25 (3) times greater than $S_0^{(N=0)}$. For this reason, the quantities $\langle \xi^N \rangle$ found in Ref. 2 are greater by a factor of 2.25 (3) (etc.) than the "perturbative" values $3/(N+1)(N+3)$, which correspond to the asymptotic[6] wave function $\varphi_\pi^{as}(x) = 6f_\pi x\bar{x}$. The coefficient of the contribution $\alpha_s \langle \bar{q}q \rangle^2$ (the most important contribution) in sum rule (5) falls off with increasing N in roughly the same way as the perturbative contribution, so that an analysis of sum rule (5) yields values for the lower moments,

$$\langle \xi^2 \rangle = 0.25 \pm 0.01; \quad \langle \xi^4 \rangle = 0.13 \pm 0.01; \quad \langle \xi^6 \rangle = 0.07 \pm 0.02, \tag{6}$$

which differ only slightly from the asymptotic values. Not surprisingly, the model-based wave function

$$\varphi_\pi^{\text{mod}}(x) = \frac{8}{\pi} f_\pi \sqrt{x\bar{x}} ,$$

which reproduces the values in (6), is also close to $\varphi_\pi^{as}(x)$. We wish to stress that the higher values for $\langle \xi^N \rangle$ in Ref. 2 are a direct consequence of the approximations $\Phi(\nu) \sim \delta(\nu), \Psi(\nu) \sim \delta'(\nu)$. For the functions Φ and Ψ which lead to the "observable" value $\lambda^2 = 0.4$ GeV^2 of the ratio $\langle \bar{q}D^2 q \rangle / \langle \bar{q}q \rangle$, the result for $\langle \xi^N \rangle$ is always close to (6).

In principle, explicit expressions for the functions Φ and Ψ could be found from the specific model (ideally, from a theory) of the QCD vacuum. A more realistic approach in practice would be based on the circumstance that sum rules analogous to those in (5) can also be found for other wave functions and also for the quark distributions in hadrons found experimentally. There is accordingly the opportunity of formulating an inverse problem: determining the vacuum distribution functions $\Phi(\nu)$ and $\Psi(\nu)$ (these distribution functions would be universal for all hadrons!) from given functions $f_{u/p}(x), f_{d/p}(x), f_{u/\pi}(x) \dots$.

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[1)] Here and below, the quark fields $u(z)$ and $d(z)$ and the gluon fields $A_\mu(z)$ are taken in the Fock-Schwinger gauge,[3] $z_\mu A_\mu(z) = 0$, in which the covariant derivatives D_μ are the same[4] as the ordinary derivatives ∂_μ.

[2)] In the derivation of QCD sum rules, it is always possible to make a Wick rotation $z_0 \to iz_0$, i.e., to assume that all the coordinates are Euclidean ($z^2 < 0$).

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