Exact two-particle *S* matrix of quantum solitons of the sine-Gordon model

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Exact and explicit formulas are constructed for the *S*-matrix elements of soliton-antisoliton scattering. These formulas agree with the perturbation theory of the Thirring massive model and with the quasiclassical expressions.

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It is known that the quantum sine-Gordon model, i.e., the model of the field $\Phi(x)$ in $(1+1)$-dimensional space-time, described by the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \Phi)^2 + \frac{m_0^2}{\beta^2}\cos(\beta\Phi) \tag{1}$$

has an infinite number of conservation laws.[1,2] This circumstance imposes stringent constraints on the scattering properties of the particles in this model. Namely, the set of particles constituting the final state of scattering and the set of their momenta coincide with the set of particles and set of momenta in the initial state, i.e., the particles can only exchange momenta in the course of scattering.[3–6]

The spectrum of the particles of model (1) consists of a soliton A, an antisoliton $\bar{A}$, and a certain number (which depends on the value of β^2) of soliton-antisoliton bound states C_n. The masses of the latter are given by the formulas[3,7]:

$$m_n = 2m\sin{}^{n\gamma}/_{16}\,, \qquad n = 1, 2, \ldots < {}^{8\pi}/_{\gamma}\,, \tag{2}$$

where m is the mass of the soliton and $\gamma = \beta^2[1-\beta^2/8\pi]^{-1}$.

The matrix elements S for two-particle $A+\bar{A}$ scattering have two components, $S_1(s)$ and $S_2(s)$ $[s=(p_1+p_2)^2$, p_1 and p_2 are the momenta of the initial particles], which describe respectively two channels of the $A+\bar{A}$ reaction: forward scattering (FS) and backward scattering (BS). $S_1(s)$ and $S_2(s)$ are analytic functions of the complex variable s on a plane with two cuts along the real axis, $s \leqslant 0$ and $s \geqslant 4m^2$.

Since the amplitudes of the $A+\bar{A}$ and $A+A$ scattering satisfy only two-particle unitarity conditions, the threshold points $s=0$ and $s=4m^2$ are root branch points of second order of the functions $S_1(s)$ and $S_2(s)$ (the branch point $s=\infty$ has, generally speaking, a logarithmic character). Therefore, if we use the variable

$$\theta = \ln\frac{s-2m^2+\sqrt{s(s-4m^2)}}{2m^2} \tag{3}$$

then the functions $S_1(\theta)$ and $S_2(\theta)$ become meromorphic functions of θ.

The transformation (3) maps the physical sheet of the s plane into a strip $0 \leqslant \mathrm{Im}\theta \leqslant \pi$, and the edges of the right-hand and left-hand cuts of the sheet are mapped on the axes $\mathrm{Im}\theta=0$ and $\mathrm{Im}\theta=\pi$, respectively. Thus, the $A+\bar{A}$ scattering (s channel) is described by the values of $S_1(\theta)$ and $S_2(\theta)$ on the semiaxis $\mathrm{Im}\theta=0$, $\mathrm{Re}\theta>0$, while the scattering $A+A$ (the u channel for FS) is described by the values of $S_1(\theta)$ on the semiaxis $\mathrm{Im}\theta=\pi$, $\mathrm{Re}\theta<0$. For BS, the u channel coincides with the s channel, so that the crossing-symmetry relation for $S_2(\theta)$ is

$$S_2(\theta) = S_2(i\pi-\theta)\,. \tag{4}$$

The unitarity conditions for the scatterings $A+\bar{A}$ and $A+A$ can be represented in the following analytic form:

$$\begin{gathered} S_1(\theta)S_1(-\theta) + S_2(\theta)S_2(-\theta) = 1, \\ S_1(\theta)S_2(-\theta) + S_1(-\theta)\,S_2(\theta) = 0, \\ S_1(\theta)\,S_1(2\pi i-\theta) = 1\,. \end{gathered} \tag{5}$$

Formula (2) means that on the segment $0<\mathrm{Im}\theta<\pi$ of the imaginary axis of the θ plane, at the points

$$\theta_n = i\pi - in\,{}^{\gamma}/_{8}\,; \qquad n = 1,2,\ldots < {}^{8\pi}/_{\gamma} \tag{6}$$

there are poles corresponding to the bound states C_n. At $\gamma=8\pi/N$, where $N=1,2,\cdots$, decay of the nth bound states place, i.e., the corresponding pole leaves the strip $0<\mathrm{Im}\theta<\pi$. Korepin and Faddeev[3] have proposed that at $\gamma=8\pi/N$ the exact expression for $S_1(\theta)$ is

$$S_1(\theta) = e^{iN\pi}\prod_{n=1}^{N}\frac{e^{\theta-i\frac{\pi n}{N}}}{e^{\theta}+e^{-i\frac{\pi n}{N}}} \tag{7}$$

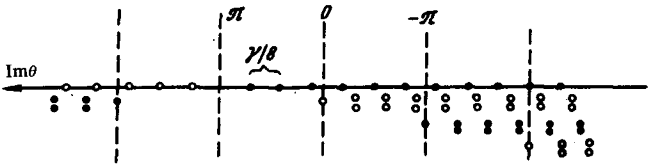

FIG. 1. Pattern of singularities of $S_1(\theta)$; the crosses designate zeros, and the points poles. Some of the zeros and poles are shifted for the sake of clarity from the imaginary axis. In reality all the singularities are at $\mathrm{Re}\theta = 0$.

This assumption is confirmed by the quasiclassical expression[8] for the function $S_2(\theta)$, which vanishes at $\gamma = 8\pi/N$ [if $S_2(\theta) \equiv 0$, then formula (7) is a consequence of (5) and (6)].

Assuming expression (7) to be exact, we shall attempt to reconstruct $S_1(\theta)$ and $S_2(\theta)$ for arbitrary γ. Since there are no resonances in the $A + \bar{A}$ channel,[7] formula (6) means that S_1 has a series of poles $\theta_n = i\pi - in\gamma/8$, $n = 1, 2, \cdots$, to ∞. At $\gamma = 8\pi/N$, there are zeros of $S_1(\theta)$ in the band $-\pi < \mathrm{Im}\theta < 0$ at points $-i\pi n/N$, $n = 1, 2, \ldots, N-1$ [see (8)]. Therefore, at arbitrary γ, the function $S_1(\theta)$ should have zeros at the points $-in'\gamma/8$, $n' = 0, 1, 2, \cdots$ to ∞, with the first zero ($n' = 0$) simple, and the remainder double. A similar analysis in the bands $-l\pi < \mathrm{Im}\theta < (l-1)\pi$, $l = 1, 2, \cdots$, and the use of condition (5), gives the picture of the zeros and poles of $S_1(\theta)$ shown in Fig. 1. The corresponding analytic expression is

$$S_1(\theta) = -\frac{i}{\pi}\,\mathrm{sh}\left(\frac{8\pi}{\gamma}\theta\right) U(\theta), \tag{8}$$

where

$$U(\theta) = \Gamma\left(\frac{8\pi}{\gamma}\right)\Gamma\left(1 + i\frac{8\theta}{\gamma}\right)\Gamma\left(1 - \frac{8\pi}{\gamma} - i\frac{8\theta}{\gamma}\right)\prod_{l=1}^{\infty}\frac{\mathrm{Re}(\theta)\,\mathrm{Re}(i\pi - \theta)}{\mathrm{Re}(0)\,\mathrm{Re}(i\pi)}, \tag{9}$$

$$\mathrm{Re}(\theta) = \frac{\Gamma\left(2l\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right)\Gamma\left(1 + 2l\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right)}{\Gamma\left((2l+1)\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right)\Gamma\left(1 + (2l-1)\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma}\right)}$$

The conditions (5) are satisfied if

$$S_2(\theta) = -\frac{1}{\pi}\sin\left(\frac{8\pi}{\gamma}\right)^2 U(\theta). \tag{10}$$

This expression satisfies automatically the crossing symmetry (4).

As $\gamma \to 0$, the functions (8) and (10) go over into the known quasiclassical expressions for the soliton S matrix.[3,9]

According to[10], the model (1) is equivalent to the Thirring massive model (TMM), the solitons being the principal fermions of the TMM. As $\gamma \to 8\pi$, formulas (8) and (10) can be expanded in powers of $2g/\pi = (8\pi/\gamma) - 1$ (g is the TMM coupling constant) and compared with the results of the TMM perturbation theory. This comparison was carried out up to order g^2 and agreement was obtained.

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