Pulsating solitons

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Pulsating solutions of the classical equations of neutral scalar fields are constructed. In the nonrelativistic approximation of quantum theory, they correspond to bound states of a large number of bosons. The energy of an N-particle bound state is calculated in the quasiclassical approximation.

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The question of the structure of bound states in field theory is directly connected with the problem of listing the soliton-like solutions of the corresponding classical equations.[1] At least in the quasiclassical approximation, each family of stable time-periodic solutions with finite energy corresponds to a set of bound states (generally speaking metastable), which are separated by a quasiclassical quantization rule. Since this question is of principal significance, and since the required analytic technique is practically nonexistent, different authors[2,3] have recently undertaken a number of numerical experiments. For example, long-lived pulsating solutions were observed for the Higgs-field

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equation; these solutions are well localized in space and are quite close to being periodic in time. Such solutions should correspond to resonances in quantum theory.

In this paper we proposed an analytic interpretation of some of the solutions of this type. We consider Lagrangians of the type

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2\phi^2}{2} - f(\phi) ,$$

with arbitrary interaction $f(\phi)$ satisfying the condition $m^2\phi^2/2 + f(\phi) \geqslant 0$, which ensures positiveness of the energy

$$H = \int\left(\frac{\pi^2}{2} + \frac{(\vec{\nabla}\phi)^2}{2} + \frac{m^2\phi^2}{2} + f(\phi)\right)d\vec{x}. \tag{1}$$

We are interested in a stable solution of the equations

$$\phi_t = \frac{\delta H}{\delta \pi}, \quad \pi_t = -\frac{\delta H}{\delta \phi}, \tag{2}$$

such that $\phi(t+T) \approx \phi(t)$ and $H[\pi, \phi] < \infty$.

There is only one known circumstance that makes it possible to guarantee the stability of a localized solution. This is the existence of an additional integral of motion, such that the proposed solution realizes a minimum of the energy at a fixed value of the integral. On the other hand, Eqs. (2) have in the general case only trivial first integrals—the momentum and the angular momentum—which have no bearing on the problem in question. Nonetheless, the integral of interest to us can be found, if not on all of phase space of the system (2), at least in a region consisting of functions of π and ϕ that vary slowly with x. The Fourier transforms of such functions are localized in a small ($\sim k_0 \ll m$) vicinity of the zero of momentum space, i.e., the field is made up of slow particles, the kinetic energy of which does not exceed $k_0^2/2m$. Since there are no spontaneous processes in classical theory, and in particle collisions the newly produced particles should likewise not have momenta larger than k_0, processes in which the number of particles changes, for example $n \longleftrightarrow n+1$, are possible only if $n > 2m^2/k_0^2$. The corresponding phase volume depends exponentially on n, so that in the indicated region of phase space the number of particles is conserved with exponential accuracy in k_0^2/m^2. It is this integral which we shall use.

It is possible to give an exact meaning to the foregoing. There exists a canonical transformation, asymptotic in the parameter $\vec{\nabla}^2/m^2$, from the variables $\pi(x)$ and $\phi(x)$ to the complex variables $\psi(x)$ and $\psi^*(x)$, such that the Hamiltonian (1) is represented in the form

$$H = \int\{F(|\psi|^2) + A(|\psi|^2)\,|\nabla\psi|^2 + [B(|\psi|^2)\,\psi^{*2}(\nabla\psi)^2 + \text{c.c.}]\}dx + O\left(\frac{\nabla^4}{m^4}\right) \tag{3}$$

and conserves the integral $N = \int|\psi|^2 dx$ ("the number of particles"). The equations of motion of the new variables are of the form

$$i\psi_t = \delta H/\delta\psi^* . \tag{4}$$

The functions F, A, and B in (3) can be calculated in explicit form from the given interaction $f(\phi)$. By considering, for example, the solutions of (2) which do not depend on x, we can easily see that the function $\xi(F)$, which is the inverse of $F(\xi)$, is given by the integral

$$\xi(F) = \frac{1}{\pi} \int \sqrt{2F - m^2\phi^2 - 2f(\phi)}\, d\phi, \tag{5}$$

where the integration is carried out between the zeros of the radicand (for the sake of argument we confine ourselves to the case of a monotonic dependence of $f'(\phi) + m^2\phi$ on ϕ). We do not need the expressions for A and B. We note only that $A(\xi) \to (2m)^{-1}$ as $\xi \to 0$.

Periodic solutions of (2) correspond to solutions of (4) in the form $\psi(\mathbf{x},t) = \phi(\mathbf{x})e^{-i\omega t}$. The function ϕ satisfies the equation

$$\omega\, \phi = \delta H / \delta \phi^* \tag{6}$$

and minimizes H at fixed N, i.e., $\delta(H - \omega N) = 0$. The requirement that H be minimal is equivalent in this case to the condition that the solution be stable.

The structure of the solutions (6) can be easily calculated at large N. In this case the principal contribution to H [Eq. (3)] is made by the term $H_0 = \int F(|\psi|^2)\, d\mathbf{x}$. Minimization of H_0 yields as a result equality of $|\psi|^2$ to n_0 inside a sphere of radius $R = (3N/4\pi n_0)^{1/3}$ and vanishing of $|\psi|^2 = 0$ outside this sphere. n_0 is then a positive root of the equation

$$\frac{d}{d\xi} \frac{F(\xi)}{\xi} = 0,$$

if such a root exists. At small ξ we have $F(\xi) = m\xi + O(\xi^2)$, but if $f(\phi)$ is a polynomial of degree $2K$, then at large ϕ we get $F \sim \xi^2(K/K+1)$ from (5); therefore a nontrivial minimum of the function $F(\xi)\xi^{-1}$ exists if $F''(\xi)|_{\xi=0} < 0$. This inequality is the condition for the existence of the solutions of interest to us. In this case $H_0 = F(n_0)n_0^{-1}N < mN$. Allowance for the remaining terms in H [Eq. (3)], which contain derivatives of ψ, leads to a "smearing" of the boundary of the sphere over a dimension of the order of $l^{-1} \sim m(1 - F(n_0)n_0^{-1}m^{-1})^{1/2}$; the corresponding contribution to the energy is proportional to the volume in which the gradients of ψ are significant, i.e., $4\pi R^2 l$. Thus, at sufficiently large N, we have

$$E(N) = \min H(\psi, \psi^*)|_N = F(n_0)\, n_0^{-1}\, N + aN^{2/3}. \tag{7}$$

In three-dimensional geometry, stable solutions of (6) exist only at $N \geq N_c$; N_c can be roughly estimated from the equality $E(N_c) = mN_c$ [Eq. (7)] which is valid at $N \gg N_c$.

So far we have dealt with solutions of classical equations. The indicated solutions have, however, a clear quantum interpretation. In the quasiclassical approximation, the quantization is invariant to canonical transformations; in terms of the variables ψ and ψ^*, on the other hand, the quantization of formula (7) consists of replacing the parameter N in this equation by integers. Equation (7) then yields the energy of the bound states of N particles ($N > N_c$). If $N \gg N_c$, this state has the structure of a Bose-liquid drop. The droplike character of the bound states of a large number of bosons in the presence of pair attraction and three-particle repulsion has already been discussed before[4] with a one-dimensional nonrelativistic model as an example.

The nonconservation of N due to the asymptotic character of the transformation π, $\phi \longleftrightarrow \psi, \psi^*$ leads to "evaporation" of the drop. The rate of evaporation is then exponentially small if $lm \gg 1$. This inequality is satisfied, for example, for interactions of the type $f(\phi) = -\lambda\phi^4 + c\phi^6$ at sufficiently small λ; in the general case this requires smallness of the pair attraction in comparison with the multiparticle repulsion.

We note in conclusion the following: At $|\psi|^2 \ll n_0$, expanding F accurate to second-order terms and replacing A in (3) by $(2m)^{-1}$, we obtain from (4)

$$i\psi_t = m\psi - \frac{1}{2m}\Delta\psi + \frac{F''(0)}{2}|\psi|^2\psi .$$

Solutions of this equation, in the form $\psi = \phi e^{-i\omega t}$, were used as a first-order approximation for pulsating solitons of Eqs. (2) in both a one-dimensional[5] and in a three-dimensional[2,3] situation. These cases, however, are different in principle. Whereas in one-dimensional geometry the solitions are stable and realize the minimum of the energy

$$H = \int\left(m|\psi|^2 + \frac{1}{2m}|\vec{\nabla}\psi|^2 + \frac{F''}{2}|\psi|^4\right)$$

at fixed N, in three-dimensional space the soliton has an exponential instability. The instability growth rate is of the order of $|\omega - m|$. The instability itself follows already from the fact that for such solutions $H > mN$, i.e., the soliton is not a bound state.[6]

The instability of three-dimensional soliton solutions with amplitude lower than a certain critical value was observed in[2] (1977) by a numerical method.

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