Multicomponent de Sitter (inflationary) stages and the generation of perturbations

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General formulas are derived for the degree of expansion of the universe and for the amplitude of adiabatic perturbations generated by the quantum fluctuations in the de Sitter (inflationary) stage which is produced by the combined action of an arbitrary number of scalar fields with different potentials and the quantum-gravitational correction ($\propto R^2$) to the Lagrangian.

We know that the following factors may be responsible for the exponential expansion of the early universe (known as the de Sitter or inflationary stage): quantum corrections, which are quadratic in the curvature tensor, to the gravitational-field equations[1] and the various scalar fields. In the second case, situations involving a phase transition[2,3] and those without it[4] are possible. The present-day supersymmetry and supergravity theories include many scalar fields, whose interaction potentials may be arbitrary to a certain extent. It is therefore useful to study the de Sitter stage which is produced by the combined action of several scalar fields with different potentials and the quantum-gravitational effects. We call such a de Sitter stage a multicomponent stage. A very simple example of this problem (a single scalar field and the quantum correction $\propto R^2$, where R is the Ricci-tensor trace) was analyzed by Kofman *et al.*[5] In this letter we derive general equations which describe the de Sitter stage when an arbitrary number of scalar fields interact with each other only in a gravitational manner [so that the potential-energy density is $V = \Sigma_n V_n(\Phi_n)$, where n is the field number] and when there is a quantum correction $\propto R^2$. We will disregard the nongravitational interaction of scalar fields with each other. An approximation of this sort is valid, on the one hand, since the nongravitational interaction cannot be much stronger than the gravitational interaction if it is assumed that perturbations (divergences from homogeneity and isotropy) generated in the de Sitter stage are not too large. The final results, on the other hand, turn out to be unusually simple and elegant in this case.

The Lagrangian of this inflationary model is

$$\mathcal{L} = \frac{1}{16\pi G}\left(-R + \frac{R^2}{6M^2}\right) + \sum_n \left(\frac{1}{2}\Phi_{n,\mu}\Phi_n^{,\mu} - V_n(\Phi_n)\right). \tag{1}$$

We assume that $a(t)$ is the scale factor of the Friedmann flat model (the space curvature can be disregarded shortly after the beginning of the de Sitter stage), and $H \equiv \dot{a}/a$. From the standpoint of practical cosmology (without considering the value of n_B/n_γ and certain exotic elementary particles like the monopoles), all the necessary information about the de Sitter stage, which the theory of this stage can provide, is (a) the extent to which the universe expands during the de Sitter stage $I = a_1/a_0$, where a_0 and a_1 are the values of $a(t)$ in the beginning ($t = t_0$) and at the end ($t = t_1$) of this stage, respectively, and (b) the spectrum and amplitude of the scalar perturbations and of the gravitational waves generated by the quantum vacuum fluctuations during the de Sitter stage.

To calculate I, we will consider a homogeneous evolution: $\Phi_n = \Phi_n(t)$. For simplicity, let us first assume that there is no quantum gravitational correction ($M = \infty$). At the quasi-de Sitter stage ($|\dot{H}| \ll H^2$), the following equations can be simplified:

$$3H^2 = 8\pi G \sum_n{}' V_n(\Phi_n), \quad \dot{H} = -4\pi G \sum_n \dot{\Phi}_n^2,$$
$$3H\dot{\Phi}_n + \frac{dV_n}{d\Phi_n} = 0, \tag{2}$$

where the prime on the sum means that it contains only those fields for which the condition $|\dot{\Phi}_n| \ll H\Phi_n$ holds [the last equation in (2) applies only to such fields]. The system of equations (2) has the particular integral

$$\ln\frac{a(t)}{a_0} = \int_{t_0}^{t} H dt = \int_{t_0}^{t} \frac{dt}{H} H^2 = \frac{8\pi G}{3} \sum_n{}' \int_{t_0}^{t} \frac{dt}{H} V_n(\Phi_n)$$
$$= -8\pi G \sum_n{}' \int_{\Phi_{n0}}^{\Phi_n} \frac{V_n(\Phi_n) d\Phi_n}{dV_n/d\Phi_n} \tag{3}$$

[in the last step, we have substituted $[-3d\Phi_n/(dV_n/d\Phi_n)]$ for dt/H in each term]. The right side of (3) is a sum of the terms, each of which depends on only a single scalar field Φ_n. Working in a similar way, for $M < \infty$ we find

$$\ln\frac{a(t)}{a_0} = -8\pi G \sum_n{}' \int_{\Phi_{n0}}^{\Phi_n} \frac{V_n(\Phi_n) d\Phi_n}{dV_n/d\Phi_n} + \frac{3}{M^2}(H_0^2 - H^2). \tag{4}$$

Comparing the expressions on the right side of (4) with those obtained in Refs. 4 and 6 for the inflationary scenarios, in which the de Sitter stage is created by a single scalar field or by quantum gravitational corrections, we obtain the first multicomponent-inflation rule

$$I \equiv a_1/a_0 = \prod_{p=1}^{n+1} I_p. \tag{5}$$

The expansion of the universe during the de Sitter multicomponent stage is equal to the formal product of the corresponding quantities for the de Sitter stages created by each component in the absence of other components.

In Eq. (5) the effect of the components on each other is seen only implicitly, through the interdependence of the quantities $\Phi_n(t_1)$ and $H(t_1)$.

The spectrum of gravitational waves generated in the de Sitter multicomponent stage is the same as the spectrum of gravitational waves generated in the single-component stage. At $k \ll aH$ this spectrum is given by the equation (taken from Ref. 7)

$$h_l^m = -\frac{\delta g_{lm}}{a^2(t)} = (2\pi)^{-3/2} \sum_j \int d^3k e_l^m \exp(i\,\mathbf{kr}) h_g(k) c_g(\mathbf{k}, j) \; ; \tag{6}$$

$$l, m = 1, 2, 3; \quad \langle c_g(\mathbf{k}, j) \rangle = 0; \langle c_g(\mathbf{k}, j), c_g(\mathbf{k}', j') \rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{jj'} ;$$

$$h_g(k) = k^{-3/2} (16\pi G H^2(t_k))^{1/2} (1 + 4H^2(t_k)/M^2)^{-1/2} ,$$

where e_l^m is the polarization tensor; $e_{lm} e^{lm} = 1$ for each state of the polarization labelled by the index $j = 1,2; k = |\mathbf{k}|; c_g(k,j)$ are the random Gaussian quantities, and t_k is determined from the equation $k = a(t)H(t)$ in the de Sitter stage. $H(t_k)$ is nearly independent of k (typically it depends on k in a logarithmic manner), so that spectrum (6) is virtually flat (it is completely flat at $H \gg M$).

At $k \ll aH$, system of equations (1) generally has $n+1$ nondecaying modes of scalar perturbations (and just as many decaying modes). Of these modes the most important one is the nondecaying adiabatic mode. Using a gauge transformation, at $k \ll aH$ we can always right the corresponding perturbed metric in the diagonal form

$$ds^2 = dt^2 - a^2(t)(1 + h(\mathbf{r}))dl^2 + O((k/aH)^2) ; \tag{7}$$

$$dl^2 = dx^2 + dy^2 + dz^2 ,$$

after the completion of the de Sitter stage. In the dust-laden stage $[a(t) \propto t^{2/3}]$, the perturbation of the density of matter would be $\delta\rho/\rho = -\eta^2 \Delta h/20 = -9t^2\Delta h/20a^2$.

In the case of other modes (constant-energy modes), in leading order in k/aH the perturbations of the metric, of the total energy density of matter, and of the nongravitational fields vanish. The nondecaying adiabatic mode, which appears in the de Sitter stage, always manages to survive to the present time (irrespective of the physics of the intermediate stages of expansion of the universe from the Planckian curvatures to the present curvature) by virtue of the causality principle. Constant-energy perturbations can survive only under special conditions (one of the components, for example, must not mix with the other components after the completion of the de Sitter stage and must remain until the present time; specifically, such a component could be the axions). We restrict the discussion here to the adiabatic mode; the constant-energy modes are discussed in Ref. 8.

The simplest (and at the same time a general) method of obtaining an expression for $h(\mathbf{r})$ is the one that was used for the derivation of Eq. (17) in Ref. 9. In a matched reference frame, at $k \ll aH$ the perturbed metric retains its diagonal form in first approximation, and the quantities h, t_0, t_1, and I become the functions of $\mathbf{r}$. In the de Sitter stage, we would have

$$ds^2 = dt^2 - a_0^2 \exp\left(2\int_{t_0}^{t} H dt\right) dl^2 , \tag{8}$$

and after its termination we would have

$$ds^2 = dt^2 - \exp\left(2\int_{t_0}^{t_1} H dt\right) a^2(t - t_1) dl^2 . \tag{9}$$

Comparing (7) and (9), we find for $t \gg t_1(\mathbf{r})$

$$h(\mathbf{r}) = 2\,\delta \ln I(\mathbf{r}) = 2\left(\sum_n \frac{\delta \ln I}{\delta \Phi_{n0}}\,\delta\Phi_n(\mathbf{r}) + \frac{\delta \ln I}{\delta R_0}\,\delta R(\mathbf{r})\right). \tag{10}$$

It follows from (2) and (3) that if there is only one scalar field, Eq. (10) becomes the equation $h = -2H\delta\Phi/\dot{\Phi}$ which was previously used in Refs. 9 and 10. Transforming to a Fourier representation and using the known quantum vacuum fluctuations $\delta\Phi_n(\mathbf{k})\delta R(\mathbf{k})$ in the de Sitter stage, we find

$$h(\mathbf{r}) = (2\pi)^{-3/2} \int d^3k \exp(i\mathbf{k}\mathbf{r})\, h(\mathbf{k}); \tag{11}$$

$$h(\mathbf{k})k^{3/2} = (8\pi G\sqrt{2}\,\sum_n{}' \frac{V_n}{dV_n/d\Phi_n} Hc_{an}(\mathbf{k}) + (24\pi GM^2)^{1/2}$$

$$\times \frac{H^2}{M^2} c_{a,n+1}(\mathbf{k}))\big|_{t=t_k};$$

$$\langle c_{ap}(\mathbf{k})\rangle = 0;\ \langle c_{ap}(\mathbf{k}),\, c_{ap'}(\mathbf{k}')\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,\delta_{pp'};\ p = 1\ldots n+1,$$

where $c_{ap}(\mathbf{k})$ are independent random Gaussian quantities. The first part of (11) is nearly independent of k, so that the spectrum is virtually flat. From (10) and (11) we find the second multicomponent-inflation rule

$$h(\mathbf{r}) = \sum_p h_p(\mathbf{r});\ \langle h^2\rangle = \sum_p \langle h_p^2\rangle. \tag{12}$$

The nondecreasing adiabatic mode which is generated in the multicomponent de Sitter stage is equal to the formal sum of the corresponding quantities for the de Sitter stages produced by each component in the absence of the other components.

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