Oscillations of the boundary between *A* and *B* phases of a superfluid ^{3}He

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The hydrodynamics of the boundary between the *A* and *B* phases of superfluid ^{3}He is analyzed. The existence of two weakly damped surface modes is predicted.

The boundary between coexisting *A* and *B* ^{3}He phases is apparently a unique example of an interface between two superfluid liquids. The peculiar behavior of this system stems from the presence of a condensate. The ^{3}He atoms in the condensate can in a thermodynamically equilibrium state tunnel through the boundary which is a potential barrier of height on the order of the energy gap Δ. The dissipative current flowing across the boundary, which is associated with the supercondensate particles, is, on the other hand, exponentially small [within $\sim\exp(-\Delta/T)$] because of an exponential decrease of the density with the temperature in the *B* phase and because of the elastic reflection of quasiparticles of energy lower than the energy gap Δ upon collision with the boundary in the *A* phase. Consequently, in contrast with the ordinary liquids, we are dealing with a two-parameter family of Goldstone transformations that conserve the energy of the system. This process first involves a uniform displacement of the interface in the case of a liquid at rest and, secondly, it involves a flow of liquid in the case of a stationary boundary. Accordingly, the oscillation spectrum of the boundary between *A* and *B* phases should have two weakly damped modes.

Let us consider the wave spectrum at $T=0$. Since the energy of the *A* phase decreases $\sim\chi H^2$ in a magnetic field and the energy of the *B* phase remains virtually the same, this situation can actually occur in $\sim 10^4$-G fields. We assume that the boundary is described by the equation $z=\zeta(x,t)$ (at equilibrium we have $z=0$), the *A* phase occupies the half-space $z>\zeta$, the *B* phase occupies the half-space $z<\zeta$, the normal to the boundary, **n**, is directed from the B phase to the A phase, and the magnetic field **H** is parallel to the $\hat{z}$ axis. We start with the linearized law of conservation[1] of matter

$$\rho_s^A(v_z^A - \dot{\zeta}) = \rho_s^B(v_z^B - \dot{\zeta}) \ , \tag{1}$$

where z are the momentum-flow components

$$p^A - p^B = \alpha_{ij}\frac{\partial^2\zeta}{\partial x_i \partial x_j} \ , \tag{2}$$

and the energy conservation law which is accurate to within second-order small terms,

$$\dot{\epsilon}_s + \frac{\partial \psi_i}{\partial x_i} = (Q_z^B - E^B\dot{\zeta}) - (Q_z^A - E^A\dot{\zeta}),$$

$$\epsilon_s = \frac{1}{2}\tilde{\alpha}_{ij}\frac{\partial\zeta}{\partial x_i}\frac{\partial\zeta}{\partial x_j} + \frac{\rho_s^{A^2}}{2I}(v_z^A - \dot{\zeta})^2 + \epsilon_s^s \ , \tag{3}$$

$$E^A = \mu^A\rho^A - p^A + \mu_B s_z^A H; \quad E^B = \mu^B\rho^B - p^B + \mu_B s_z^B H; \quad \psi_i = -\tilde{\alpha}_{ij}\frac{\partial\zeta}{\partial x_j}\dot{\zeta},$$

$$Q_z^A = \mu^A\rho^A v_z^A + \frac{\hbar\rho_s^A}{4m^2}(\hbar\dot{\theta}_z^A + 2\mu_B H)\partial_z\theta_z^A;$$

$$Q_z^B = \mu^B\rho^B v_z^B + \frac{\hbar\rho_s^B}{10m^2}(\hbar\dot{\theta}_z^B + 2\mu_B H)\partial_z\theta_z^B .$$

Here all the quantities corresponding to the A and B phases are denoted by appropriate indices; ρ, p, μ, μ_B, E, and **Q** are the density, pressure, chemical potential, nuclear (Bohr) magneton, energy, and bulk energy flux; the second terms in E and **Q** are the energy and energy flux associated with the spin variables; and ϵ_s and ψ are the surface energy and surface energy flux. The indices i and j assume the values 1 and 2. In the second term in ϵ_s, we took into account the Josephson energy in the quadratic approximation. Since the Josephson energy is an internal energy, it is determined in a coordinate system associated with the moving boundary. The third term ϵ_s is the surface energy associated with the spin variables. The equilibrium boundary condition[2] specifies the spin coordinates of the A and B phases in such a way that $(\mathbf{n}\cdot R(\omega,\theta_L)\cdot\mathbf{d}) = 1$. $(R(\omega,\theta_L))$ is the matrix of the rotation of the spin-space relative to the orbital space around ω through the angle $\theta_L = \arccos(-\frac{1}{4})$, and **d** is the spin-anisotropy axis of the A phase. From the symmetry we see that $\epsilon_s^s = \beta(1 - \mathbf{n}\cdot R(\omega,\theta_L)\cdot\mathbf{d})$, where $\beta \sim g_D\zeta_0$. In a magnetic field $\mathbf{H}\|\hat{z}$ only the s_z spin component is conserved. The quantity which is dynamically coupled to s_z is the rotation angle of the order parameter, θ_z. Describing ϵ_s^s in terms of small increments and singling out the part connected with θ_z, we find $\epsilon_s^s = (\beta/2)(\theta_z^A - \theta_z^B)^2$.

Let us add to the system of equations (1)–(3) the conditions under which the z components of the spin currents are conserved at the boundary between the A and B phases[1]:

$$\frac{\hbar}{2m^2}\rho_s^A(\partial_z\theta_z^A) - \frac{\chi^A H\dot{\zeta}}{\mu_B} = \frac{\beta}{\hbar}(\theta_z^A - \theta_z^B),$$

$$\frac{\hbar}{5m^2}\rho_s^B(\partial_z\theta_z^B) - \frac{\chi^B H\dot{\zeta}}{\mu_B} = \frac{\beta}{\hbar}(\theta_z^A - \theta_z^B) . \qquad (4)$$

Here χ^A and χ^B are the susceptibilities of the A and B phases. The first terms on the right sides of these equations are the bulk spin currents and the second terms are the equilibrium magnetization. The left side of the equations is the "Josephson" spin current that flows through the boundary. The liquid-crystal anisotropy in Eqs. (1)–(3) reduces to a redefinition of the surface energy, α_0, in terms of the surface "rigidity". $\widetilde{\alpha}_{ij} = \alpha_0\delta_{ij} + \alpha_1 l_i l_j$ $\mathbf{l}$ is the orbital-anisotropy vector of the A phase. Equation (3) can be transformed with the help of (1) and (2) to

$$\rho_s^A(v_s^A - \dot{\zeta})\left\{\frac{\rho_s^A}{I}(\dot{v}_z^A - \ddot{\zeta}) - (\mu^B - \mu^A)\right\} + \frac{\hbar H\dot{\zeta}}{2\mu_B}(\chi^A\dot{\theta}_z^A - \chi^B\dot{\theta}_z^B) = 0. \qquad (5)$$

Let us consider the equations in the A and B phases. We see from their solution that the velocity of the modes which we are seeking is much lower than the velocity of sound. This circumstance makes it possible to use the equations for an incompressible liquid. Introducing the velocity potential $v_i^{A,B} = (\hbar/2m)(\partial\varphi^{A,B}/\partial x_i)$, we find the Laplace equation $\Delta\varphi^{A,B} = 0$, whose solution is $\varphi^{A,B} = \widetilde{\varphi}^{A,B}\exp\{\mp kz + ik_jx_j - i\omega t\}$. The difference between the superfluid densities, $\delta\rho_s$, of the A and B phases is determined by the different structures of the order parameter and at $T=0$ is small since $\delta\rho_s/\rho \sim (\Delta/\epsilon_F)^2$. Setting $\rho_s^A = \rho_s^B = \rho$, from (1) we find $v_z^A = v_z^B$ and $\widetilde{\varphi}^A = -\widetilde{\varphi}^B$.

We seek the boundary oscillations in the form $\zeta = \widetilde{\zeta}\exp\{ik_jx_j - i\omega t\}$ and since $\delta p^{A,B} = \rho\delta\mu^{A,B} = -(\hbar/2m)\rho(\partial\varphi^{A,B}/\partial t)$, we have a Fourier transform of the conservation of the momentum flow (2)

$$i\omega\frac{\hbar\rho}{m}\widetilde{\varphi} = -\widetilde{\alpha}_{ij}k_ik_j\widetilde{\zeta}.$$

The order parameter with $\boldsymbol{\omega}\|\mathbf{H}$, which is at equilibrium in the B phase, is restored over a distance of the magnetic length ξ_m in fields $>10^4$G; here $\xi_m < \xi_D$. Since the spin oscillations in both phases are localized over a distance $\sim\xi_D$, we will use the solution of the equations of the spin dynamics for a uniform texture. For the oscillations which depend on the coordinates and on the time in accordance with $\theta_z^{A,B} = \widetilde{\theta}_z^{A,B}$ $\exp\{\mp kz - ik_jx_j - i\omega t\}$ we thus find $k_z^A = \Omega^A/c_A$; $k_z^B = \Omega^B/c^B$ to within the wave vectors $k\sim 10^5$ cm^{-1}. The Josephson spin current on the right side of the equations in (4) is, in accordance with $g_D\xi_0\mu_B/\hbar\chi Hc \sim 10^{-5}$, small at fields $\sim 10^4$ G in comparison with the terms on the left side of these equations. This circumstance suggests that there is a coupling between the spin-oscillation amplitude and the boundary velocity:

$$\tilde{\theta}_z^A = \frac{2m^2\chi_A c_A H}{\hbar\rho\mu_B\Omega_A}\ ; \quad \tilde{\theta}_z^B = \frac{5m^2\chi_B c_B H}{\hbar\rho\mu_B\Omega_B}\ .$$

Collecting all the boundary conditions we have written out and substituting them into the energy conservation law in Eq. (5), we find the dispersion equation

$$\left(\omega^2 - \frac{\tilde{\alpha}_{ij}k_ik_jk}{2\rho} - \frac{I\tilde{\alpha}_{ij}k_ik_j}{\rho^2}\right)\left(\omega^2 - \frac{\tilde{\alpha}_{ij}k_ik_jk}{2\rho}\right) + \frac{Im^2\xi_D H^2}{4\rho^3\mu_B^2}(\chi_A^2 - \chi_B^2)\omega^4 = 0. \qquad (6)$$

The dispersion equation gives two oscillation modes. Allowing for the fact that the third term in the first parentheses is large in comparison with the second term, we finally find

$$\omega_1^2 = \frac{\tilde{\alpha}_{ij}k_ik_jk}{2\rho}\ ; \quad \omega_2^2 = \frac{I\tilde{\alpha}_{ij}k_ik_j}{\rho^2}\left\{1 + \frac{Im^2\xi_D H^2}{4\rho^3\mu_B^2}(\chi_A^2 - \chi_B^2)\right\}^{-1} \qquad (7)$$

for the dominant terms in k in the dispersion laws. The first mode is an "ordinary" wave at the boundary between two liquids and the second mode is an analog of the plasma oscillations in the distributed Josephson structures. To estimate the velocity of the second mode, we will use a standard expression for the Josephson constant in the BCS theory.[3] We have $I \sim (m^2\epsilon_F^2/\hbar^2V^2)\Delta$, where $V \sim \Delta\xi_0$ is the effective potential barrier. We thus find $I \sim 10^5$ g/cm^4, $\alpha \sim (\Delta^2/\epsilon_F)n\xi_0 \sim 10^{-5}$ erg/cm^2, and $c \sim (\alpha I)^{1/2}/\rho \sim 10$ cm/s. The expression in the braces is estimated to be $\{1 + 10^{-8}H^2\}^{-1}$.

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[1]W. F. Brinkman and M. C. Cross, in: Progress in Low Temperature Physics (Ed. D. F. Brewer-Amsterdam, North-Holland, 1978, Ch. 7, p. 105; V. P. Mineev, Usp. Fiz. Nauk **139**, 301 (1983). G. E. Volovik, Usp. Fiz. Nauk **143**, 73 (1984) [Sov. Phys. Uspekhi **27**, 363 (1984)].
[2]M. C. Cross, Kvantovye zhidkosti i kristally (Quantum Liquids and Crystals), (Russ. transl. Mir, Moscow, 1979, p. 96, a collection of papers, edited by A. S. Borovik-Romanov).
[3]I. O. Kulik and I. K. Janson, Effekt Dzhozefsona v sverkhprovodyashchikh tunnel'nykh strukturakh (The Josephson Effect in Superconducting Tunnel Structures), Nauka, Moscow, 1970.

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