Symmetry of vortex structures, axial currents, and Kelvin waves in superfluid ^{3}He

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A symmetry classification is offered for vortex structures in ^{3}He. Kelvin waves are analyzed for structures which allow axial currents. If the vessel is rotating at a low velocity $\vec{\Omega}$, and the wavelength is given, the oscillation frequency is a square-root function of Ω, while the functional dependence is linear for structures without axial currents (in particular, in ^{4}He).

An approach which has proved successful in solving problems of the hydrodynamics of a rotating superfluid is based on the hydrodynamic equations averaged over scale lengths considerably greater than the period of the vortex structure. This theory ("macroscopic hydrodynamics") derives from studies of ^{4}He by Hall[1] and Bekarevich and Khalatnikov.[2] It has been used to analyze vortex waves propagating along vortices in ^{4}He (Kelvin waves).[3] Macroscopic hydrodynamics was subsequently generalized to the case of waves propagating perpendicular to the vortices (Tkachenko waves) or at some angle from them (see Refs. 4 and 5 and the bibliographies there).

The macroscopic hydrodynamics of a rotating superfluid can also be used to describe superfluid phases of ^{3}He. In ^{3}He, however, the number of possible equilibrium structures that must be dealt with is far larger than in ^{4}He, where the rotation always leads to the formation of a triangular lattice of singular vortex lines. In particular, in the A phase there may be periodic nonsingular vortex structures with a continuous distribution of the vorticity of the superfluid velocity; the study of these structures began with the paper by Volovik and Kopnin.[6]

In the present letter we offer a system of linear equations of macroscopic hydrodynamics for vortex structures in ^{3}He; we classify the possible vortex structures by symmetry; and we use the new theory to analyze the spectrum of Kelvin oscillations for certain symmetry classes.

We begin with an expression for the energy density of the liquid in the harmonic approximation in a coordinate system which is moving at the normal velocity:

$$E = (\rho_s)_{ij} \frac{v_{si} v_{sj}}{2} + \lambda_{ijkl}\, u_{ij}\, u_{kl} + \gamma_{ijk}\, v_i u_{jk} + E_0 \,. \qquad (1)$$

Here we have singled out the part of the energy which is associated with the order parameter; $\mathbf{v}_s$ is the superfluid velocity, averaged over a cell of the vortex lattice; the $u_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$ are the components of the strain tensor; and $\mathbf{u}$ is the displacement of the vortex structure, which has components in the plane perpendicular to the rotation axis (i.e., $u_z = 0$). There is no cross term containing $\mathbf{v}_s$ and $\mathbf{u}$ in ^{4}He. There is a component of the superfluid current associated with this term:

$$\lambda_i = \frac{\delta E}{\delta v_{si}} = (\rho_s)_{ij} v_{sj} + \gamma_{ijk} u_{jk} . \qquad (2)$$

We now write a system of hydrodynamic equations for a regime with a fixed normal component, with $v_n = 0$. This regime is easy to arrange in ^{3}He because of the high viscosity in that case; furthermore, even in ^{4}He the Kelvin oscillations which we will be discussing below do not entrain the normal component in any significant way. Here is the system of equations:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{\lambda} = 0 \qquad (3)$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + [2\boldsymbol{\Omega}, \mathbf{v}_L] = -\nabla\mu \qquad (4)$$

$$\mathbf{v}_L = \frac{\partial \mathbf{u}}{\partial t} = \alpha_{\alpha\beta} (\vec{\lambda} + [2\vec{\Omega}, \mathbf{F}]/\Omega^2)_\beta \quad , \qquad (5)$$

where $\alpha_{\alpha\beta}$ is a 2×2 matrix which couples the vectors in the XY plane, and $\mathbf{F} = -(\delta E/\delta\mathbf{u})$ is the elastic force acting on vortices.

Among the dissipative terms, the system of equations retains only the mutual friction, which is represented by the components of the tensor $\alpha_{\alpha\beta}$.

The hydrodynamic equations are thus written in terms of the same variables as in the case of ^{4}He (Ref. 4), although the basic hydrodynamic theory for ^{3}He is usually expressed in terms of other variables, e.g., $\mathbf{v}_s$ and $\mathbf{l}$ for the A phase. It is obvious, however, that the deviations of $\mathbf{l}$ from the equilibrium structure constitute a functional of the deformations of the structure which figure in our theory. In particular, the $\vec{\lambda}$ component of the current [see (2)] associated with the cross term γ results from an orbital component of the current, $\mathbf{l}(\mathbf{l}\ \mathrm{curl}\ \mathbf{l})$, after the vortex structure is averaged over the cell.

For a symmetry classification of periodic vortex structures we use magnetic groups, since the vortex structures are spatial distributions of the currents and of moments associated with these currents, which are similar in terms of symmetry to the magnetic moments in magnetic crystals. In macroscopic hydrodynamics there is no need to examine the complete symmetry group of vortex structures, since translations are identity transformations for a macroscopic description. We thus find a classification of vortex structures on the basis of magnetic crystal classes in which we consider only point groups supplemented with the transformation of time reversal.[7] Not all of the magnetic classes, however, are possible for two-dimensional vortex structures. The rotation axis (the vertical $\mathbf{z}$ axis) is a special axis, so that time reversal R, reflection in the vertical plane (σ_v or σ_d), and a twofold horizontal axis U_2 are not possible as individual symmetry elements, but they may be found in the combinations $R\sigma_v$, $R\sigma_d$, or RU_2. For simplicity, we will not write out the classes that contain the symmetry element σ_h: reflection in the horizontal plane. In ^{3}He-A, for example, such classes correspond to trivial vortex structures with a uniform $\mathbf{l}$, directed along the rotation axis.

We now write all 21 symmetry classes which are possible for vortex structures in ^3He (under our reservation regarding σ_h) (the classes placed in braces do not differ for cases of trivial vortex structures):

The rectangular system $\{C_{1v}(C_1); D_1(C_1)\}$; $\{C_{2v}(C_2); D_2(C_2)\ D_{1d}(C_i)\}$.

The square system $\{C_4; S_4\}$; $\{C_{4v}(C_4); D_4(C_4); D_{2d}(S_4)\}$.

The hexagonal system C_3; $\{C_6; S_6\}$, $\{C_{3v}(C_3); D_3(C_3)\}$; $\{C_{6v}(C_6);$ $D_6(C_6); D_{3d}(S_6)\}$.

We now consider Kelvin oscillations for a symmetry which allows currents of orbital origin along the **z** axis in an equilibrium vortex structure. This property is exhibited by the structures of the classes C_n, D_n (C_n). We restrict the analysis to the highest-symmetry classes $(n > 2)$, for which

$$(\rho_s)_{ij} = \rho_s \delta_{ij} \; ; \quad \alpha_{\alpha\beta} = \frac{1}{\rho_s}\left(1 - \frac{B'\rho_n}{2\rho}\right)\delta_{\alpha\beta} + \frac{\rho_n B}{2\rho\rho_s}\epsilon_{\alpha\beta z} \; ; \quad \gamma_{zij} = \gamma\delta_{ij} + \tilde{\gamma}\epsilon_{ijz} \, .$$

Solving Eqs. (3), (4), and (5) for a plane wave, $v_s \sim v_L \sim e^{i(pz - \omega t)}$), we find the spectrum of Kelvin waves:

$$\omega = \pm\left(1 - \frac{B'\rho_n}{2\rho} \mp i\frac{B\rho_n}{2\rho}\right)\left(\nu_s p^2 + 2\Omega + \gamma p\right). \tag{6}$$

Terms linear in p, which are not present in the case of ^4He, thus appear in the spectrum of Kelvin waves. In the case of a nonsingular vortex structure in ^3He-A, the only scale dimension is the period $(b \sim \Omega^{-1/2})$ of the vortex structure, so that the frequency in (6) must be a function of the dimensionless parameter pb. This conclusion means that we have $\gamma \sim \sqrt{\Omega}$. In Hall's experiments[1] with an oscillating stack of disks in ^4He, Kelvin-wave resonances were observed at fixed values of p. The appearance of terms γp in the spectrum means that the frequencies of the Hall resonances at small values of Ω must increase in proportion to $\sqrt{\Omega}$, rather than Ω, as in ^4He. It thus follows that it is possible to draw conclusions about the type of vortex structure from the data of such experiments. In particular, it is possible to distinguish between lattices of radially hyperbolic vortex pairs (v vortices) and lattices of circularly hyperbolic vortex pairs (w vortices).[8] The former belong to class C_{1v} (C_1), while the latter belong to class $D_1(C_1)$; the axial currents and terms $\sim \gamma p$ are thus permissible only for the latter lattice.

An observation of Kelvin-wave resonances in ^3He will require rather low temperatures, since the values found experimentally for the parameter B near T_c are quite large.[9]

This modification of the Kelvin-oscillation spectrum due to axial currents may also occur in the B phase. A symmetry analysis of vortices in the B phase[10] shows that such currents are possible for w and uvw vortices.

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