An exact solution of a two-dimensional, non-Baxter-type vertex model

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An exact solution of a two-dimensional vertex model that satisfies the ice rule is obtained in an external electric field. The examined model, in contrast to the generalized Baxter model, is formed by two types of alternating lattice points.

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In statistical mechanics there are a whole series of nontrivial, two-dimensional models of interacting particles that have an exact solution. Baxter showed in a recent paper[1] that all of these models can be considered as special cases of the exactly solved, generalized eight-vertex model.[2] The generalized model was defined by Baxter on a plane lattice with the coordination number 4, where vertex factors a_j,b_j,c_j, and d_j at each lattice point satisfy the conditions

$$(a_j^2 + b_j^2 - c_j^2 - d_j^2)/a_j b_j = C_1, \quad c_j d_j / a_j b_j = C_2, \tag{1}$$

where C_1 and C_2 are constants that are independent of the number j of the junction point.

In this paper we analyze a system formed by two interacting eight-vertex models and find its solution for a special case which does not explicitly reduce to a Baxter solution defined by the relations (1).

We shall examine a lattice of $2M$ rows and $2N$ columns (see Fig. 1). The lattice consists of two sublattices that are formed by the solid and dashed lines, respectively. In this case two types of lattice points are formed due to intersection of similar and dissimilar lines. We shall place an arrow on each edge of the lattice that points toward one of the lattice points. Let us assume that there are eight standard configurations[2] at points of the first type and assign to them the vertex statistical weights

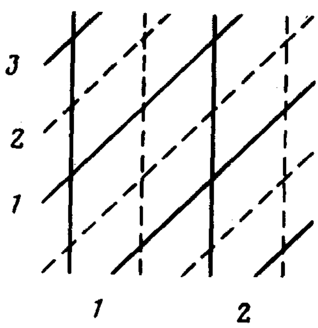

FIG. 1.

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c, \quad w_7 = w_8 = d. \tag{2}$$

Let us assume that these are only the first four configurations at the second type points

$$w_1' = w_2' = g, \quad w_3' = w_4' = h. \tag{3}$$

We shall restrict ourselves to the analysis of the special case corresponding to a system of two plane ferroelectrics ($d = 0$) that interact with each other with a force $\alpha = 2\ln(g/h)$ with the additional condition $a^2 + b^2 - c^2 = 0$. This case corresponds to a model that is not explicitly Baxter type, since the parameter C_1 has two different values at different types of lattice points.

A standard calculation[2–4] of the free energy f of the examined system in an external electric field reduces to determining the largest eigenvalue $\Lambda_{\max}$ of the transition matrix T

$$-\beta f = \frac{1}{4} \max_{y_1, y_2}{}' \lim_{N \to \infty} \left\{ \frac{1}{N} \ln [\Lambda_{max}(m, n)] + E_1 y_1 + E_2 y_2 \right\},$$

$$y_1 = 2(1 - m/N), \; y_2 = 2[1 - (n - m)/N], \tag{4}$$

where E_1 and E_2 are the external electric fields applied to the first and second sublattices, respectively, and m ($n - m$) is the number of arrows directed downward in some odd-numbered row of edges that belong to the first (second) sublattice. The transition matrix T consists of individual $T(m,n)$ units corresponding to different values of m and n. Let us assume that $\Psi_{\tau_1 \ldots \tau_n}(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n)$ is the amplitude of the odd row when the downward-directed arrows are located at the edges $(x_1, \tau_1), \ldots, (x_m, \tau_m)$ of the first sublattice and the edges $(x_{m+1} \tau_{m+1}), \ldots, (x_n, \tau_n)$ of the second sublattice, $\tau_i = 1,2$, respectively, depending on whether the ith edge is tilted or vertical. The transition matrix connects the amplitude of the two, successive, odd-numbered rows

$$\Lambda(m, n) \Psi_{\vec{\tau}}(\mathbf{x}) = \sum_{\{\mathbf{x}', \vec{\tau}'\}} T(m, n)_{\vec{\tau}\vec{\tau}'}(\mathbf{x}, \mathbf{x}') \Psi_{\vec{\tau}'}(\mathbf{x}'). \tag{5}$$

Let us divide the region in which $\Psi_\tau(\mathbf{x})$ is defined into separate subregions

$$1 \lesssim x_{Q_1} \lesssim x_{Q_2} \lesssim \cdots \lesssim x_{Q_n} \lesssim N, \tag{6}$$

where $[Q_1, \ldots, Q_n]$ is a permutation of the numbers 1, 2,..., n. The equal sign between x_{Q_i} and $x_{Q_{i+1}}$ is possible in Eq. (6) only in the following cases: (1) $Q_i \leqslant m, Q_{i+1} > m$; (2) $Q_i \leqslant m, Q_{i+1} \leqslant m, \tau_{Q_i} = 1, \tau_{Q_{i+1}} = 2$; (3) $Q_i > m, Q_{i+1} > m, \tau_{Q_i} = 1, \tau_{Q_{i+1}} = 2$. In each of the subregions (6) we shall look for the amplitude $\Psi_\tau(x)$ in the form of a generalized Bethe equation[5,6]

$$\Psi_{r_1 \ldots r_n}(x_1, \ldots, x_n) = \sum_P A^{\Delta Q_1 \ldots \Delta Q_n}_{P_1 \ldots P_n} \prod_{j=1}^{n} \exp\left[\left(\frac{1}{2} \Delta_{Q_j} - 1\right) \times k_{P_j}\right] \Psi_{r_{Q_j}}^{(\nu_{P_j}, k_{P_j})}(x_{Q_j}), \tag{7}$$

where the summation is carried out over all permutations $[P_1,\ldots,P_n]$ of the numbers 1, $2,\ldots,n$,

$$\Psi_{1,2}^{(\nu_P, k_P)}(x_Q) = \exp\left[i k_P x_Q \mp \frac{1}{2}\left(M_P + \frac{1}{2} k_P\right)\right],$$

$$\nu_P = \pm 1,$$

$$M_P = \arcsin[\bar{b} \sin(k/2)], \ \nu_P \cos M_P \geqslant 0, \ \bar{b} = b/c,$$

$\Delta_Q = 1,2$, respectively, for $Q \leqslant m$ and $Q > m$.

Equation (7) defines the solution of Eqs. (5), if the $A_P^{\Delta_Q}$ coefficients satisfy the conditions

$$A^{\Delta_1 \ldots \Delta_i \Delta_{i+1} \ldots \Delta_n}_{P_1 \ldots P_i P_{i+1} \ldots P_n}$$

$$= \sum_{\alpha,\beta=1}^{2} S^{\Delta_i \Delta_{i+1}}_{\alpha\beta}(M_{P_i} - M_{P_{i+1}}) A^{\Delta_1 \ldots \alpha\beta \ldots \Delta_n}_{P_1 \ldots P_{i+1} P_i \ldots P_n}, \tag{8}$$

$$A^{\Delta_1 \ldots \Delta_n}_{P_1 \ldots P_n} = A^{\Delta_2 \ldots \Delta_n \Delta_1}_{P_2 \ldots P_n P_1} \exp(i k_{P_1} N),$$

where S is an exact, two-particle S matrix

$$S^{\lambda\alpha}_{\alpha'\lambda'} = \sum_{k=1}^{4} \omega^k \sigma^k_{\alpha\alpha'} \sigma^k_{\lambda\lambda'},$$

$$\omega^1 = \omega^2 = -i \,\mathrm{sh}\, \alpha / 2 \sin(M + i\alpha), \ \omega^{3,4} = -\frac{1}{2}\left[1 \pm \sin M / \sin(M + i\alpha)\right]. \tag{9}$$

Proof of this assertion together with a more detailed study of the given model, will be published later.

The consistency of Eqs. (8) is guaranteed by the fact that the S matrix satisfies the Baxter-Yang relation.[4,6] The values of k_j, in Eqs. (8) in terms of which the eigenvalues of the T(m,n) matrix are expressed, are unknown. The solution of Eqs. (8), which can be found by iterative use of the generalized Bethe equation[6] reduces to the solution of the following system of nonlinear equations

$$k_j N = 2\pi I_j + \sum_{\beta=1}^{m} \phi(M_j - \Lambda_\beta),\quad j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} \phi(\Lambda_\beta - M_j) = 2\pi J_\beta + \sum_{\gamma=1}^{m} \Phi(\Lambda_\beta - \Lambda_\gamma),$$

$$\beta = 1, 2, \ldots, m, \tag{10}$$

$$\exp[i\Phi(M)] = -\sin(M - ia)/\sin(M + ia),\quad a' = a/2,$$

$$\exp[i\phi(M)] = -\sin(M - ia')/\sin(M + ia'),$$

where I_j are half-integers for the even n and J_β are integers (half-integers) for the odd (even) m.

Equations (10) make it possible, in principle, to determine the entire spectrum of the transition matrix and hence to investigate all the static properties of the examined model.

In the limit $N \to \infty$ Eqs. (10) are replaced in the standard manner[5-7] by linear integral equations. In the case $n = 2m = 2N$ corresponding to Λ_{max}, these equations, which are solved by means of a Fourier transformation, give rise to the following expression for the free energy in the "physical" region ($\bar{b} < 1$)

$$-\beta f = \frac{1}{2} \ln(ah) - \text{sign}(a) \int_0^{\pi} \ln\left[\cos p + \sqrt{(a/b)^2 + \cos^2 p}\right] v(p)\, dp, \tag{11}$$

where

$$v(p) = (2\pi)^{-1}\left(\frac{dL}{dp}\right)\left[1 + \sum_{l=1} v_l \cos(2lL)\right],$$

$$v_l = \frac{4\exp(-n|a|)}{\pi\, \text{ch}(na)} \int_0^{\pi/2} \cos(2lL)\, dp,$$

$$L = L(p) = \arcsin(\bar{b}\sin p),\quad -\frac{\pi}{2} \leq L(p) \leq \frac{\pi}{2}.$$

The first order with respect to $\bar{b}$ in Eq. (11) corresponds to the ground-state energy of the one-dimensional Hubbard model.[7]

In contrast to the previously studied models that satisfy the ice rule,[1–3] this model has a first-order phase transition with respect to the interaction parameter $\alpha(\alpha_{cr} = 0)$.

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