

Branching processes and multiplicity distributions in QCD jets

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Branching processes with two types of particles were used to investigate the multiplicity distributions in QCD jets. Solutions of the evolution equations for the process were found and the KNO scaling limits were investigated.

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The concept of hadron jets like that of a branching process (BP), which was developed in the investigation of the scale-invariant, quantum field theory,¹ was used in a phenomenological approach to the e^+e^- annihilation process.² These concepts have recently been further developed in terms of the evolution equations in asymptotically free theories.³ The probability interpretation of the principal logarithm approximation in quantum chromodynamics (QCD)⁴ makes it possible to treat these equations as equations for the branching process: the branching process can be represented as a sequential decay of a parton of mass $\sqrt{Q^2}$ that produces a jet, and the parton can also be changed due to emission of other partons. The virtual masses of partons (quarks and gluons) turn out to be ordered on the tree diagrams, so that each "parent" parton is always farther from the mass surface than its "offspring." This makes it possible to

relate the time variable t of the branching process to the mass of the initial ($\sqrt{Q^2}$) and "running" ($\sqrt{k^2}$) partons

$$t = 6[11N_c - 2N_f]^{-1} \ln \left[\frac{\ln Q^2}{\ln k^2} \right]$$

in a theory with N_c colors and N_f flavors. The branching process is terminated at the "time" the partons are near their mass surfaces ($k^2 \approx Q_0^2$) and their hadronization process must be started.

We shall examine a branching process with two types of particles: quarks and gluons. The development of the branching process with time is completely determined by the probabilities of the transitions $P_{n_1, n_2}^i(t)$ to the states with n_1 quarks and n_2 gluons in the quark ($i = 1$) and gluon ($i = 2$) jets. We introduce the corresponding generating functions

$$F_i(t, z_1, z_2) = \sum_{n_1, n_2} P_{n_1, n_2}^i(t) z_1^{n_1} z_2^{n_2},$$

which contain all the necessary information about the branching process. The evolution equations for the branching process, which are written for the generating functions, have the form⁵

$$\dot{F}_i = \sum_{n_1, n_2} \lambda_{n_1, n_2}^i (F_1)^{n_1} (F_2)^{n_2} \quad (1)$$

with the initial conditions

$$F_i(0, z_1, z_2) = z_i. \quad (2)$$

The values λ^i are the matrices of the infinitesimal transitions

$P_{n_1, n_2}^i(t) = \delta_{n_1, n_2}^i + \lambda_{n_1, n_2}^i t + O(t^2)$; $\delta_{10}^1 = \delta_{01}^2 = 1$, the remaining are $\delta_{n_1, n_2}^i = 0$). It follows from the normalization condition and $P_{n_1, n_2}^i(t)$ that

$$\sum_{n_1, n_2} \lambda_{n_1, n_2}^i = 0 \quad (3)$$

where $\lambda_{10}^1 < 0, \lambda_{01}^2 < 0$, and the rest of the $\lambda_{n_1, n_2}^i \geq 0$ have the meaning of the probability density of the corresponding transitions.

The nonvanishing probability densities of the elementary transitions in the principal-logarithm approximation in QCD are: $\lambda_{11}^1 \equiv \lambda, \lambda_{20}^2 \equiv \gamma_1, \lambda_{02}^2 \equiv \gamma_2$. These values are the fundamental parameters of the branching process; The problems associated with the calculation of these values in QCD will not be examined here.

Equations (1) allowance for Eq. (3) have the following form:

$$\begin{aligned} \dot{F}_1 &= -\lambda F_1 + \lambda F_1 F_2, \\ \dot{F}_2 &= -(\gamma_1 + \gamma_2) F_2 + \gamma_1 F_1^2 + \gamma_2 F_2^2 \end{aligned} \quad (4)$$

Equations like (4) were obtained in Ref. 6 on the basis of the so-called "jet calculation rules", where the solution of the simplified Eqs. (4) with $\gamma_1 = 0$ was analyzed. From a physical viewpoint, this case represents the absence of quark generation in the branching process ($n_1 \equiv 1$ in the quark jet and $n_1 \equiv 0$ in the gluon jet) and, therefore, is of little interest. For arbitrary values of λ , γ_1 , and γ_2 the solutions of Eqs. (4) cannot be found in quadratures. Therefore, it is a pleasant surprise to be able to obtain an explicit analytical solution of (4) in the physically interesting case of $\lambda/\gamma_2 = \frac{1}{2}$. The λ/γ_2 ratio in QCD can be represented in the form

$$\lambda/\gamma_2 = 1/2 - 1/2N_c^2 \quad (\lambda/\gamma_2 = 4/9 \text{ for } N_c = 3, \quad \lambda/\gamma_2 \approx 1/2 \text{ for } N_c^2 \gg 1).$$

Eliminating F_2 in Eqs. (4), we obtain a nonlinear, second-order equation for F_1 , which can be reduced, by means of the substitution $F_1(t) = [f(t)]^{-\lambda/\gamma_2}$, to the linear derivative form

$$\ddot{f} + (\gamma_1 + \gamma_2) \dot{f} - \gamma_1 \gamma_2 f (1 - f^{-2\lambda/\gamma_2}) = 0. \quad (5)$$

When $\lambda/\gamma_2 = \frac{1}{2}$ Eq. (5) is completely linearized. Thus we obtain

$$F_1 = [A e^{\gamma_2 t} + B e^{-\gamma_1 t} + 1]^{-1/2}, \quad F_2 = 1 - F_1^2 \left[A e^{\gamma_2 t} - \frac{\gamma_1}{\gamma_2} B e^{-\gamma_1 t} \right]. \quad (6)$$

The integration constants A and B in Eq. (6) are functions of z_1 and z_2 , which can be determined by basic calculations from the initial conditions (2)

$$A = (\gamma_1 + \gamma_2)^{-1} z_1^{-2} [\gamma_1 (1 - z_1)^2 + \gamma_2 (1 - z_2)], \quad B = (\gamma_1 + \gamma_2)^{-1} z_1^{-2} \gamma_2 (z_2 - z_1^2).$$

The average number \bar{n}_j^i of partons in the jets at large multiplicities are (the superscript represents the jet and the subscript denotes the parton):

$$\bar{n}_j^i = \left. \frac{\partial F_i}{\partial z_j} \right|_{z_1 = z_2 = 1} = \frac{i}{j} \frac{\gamma_j}{\gamma_1 + \gamma_2} e^{\gamma_2 t}. \quad (7)$$

Relations (7) show that the number of each type of parton in the gluon jets is twice as large as in the quark jets. We can also find

$$\overline{n_j^i n_k^i} = \frac{\partial}{\partial z_j} z_k \left. \frac{\partial F_i}{\partial z_k} \right|_{z_1 = z_2 = 1} = (4 - i) \bar{n}_j^i \bar{n}_k^i \quad \text{or} \quad \frac{\overline{n_j^i n_k^i} - \bar{n}_j^i \bar{n}_k^i}{\bar{n}_j^i \bar{n}_k^i}. \quad (8)$$

The distributions of quarks $P^i(n_1)$ and gluons $P^i(n_2)$ in both types of jets and the KNO functions $\psi(x)$ corresponding to them are determined in the form ($x \equiv n_j/\bar{n}_j^i$):

$$P^i(n_1) = \sum_{n_2} P_{n_1 n_2}^i, \quad P^i(n_2) = \sum_{n_1} P_{n_1 n_2}^i; \quad \Psi_j^i(x) = \lim_{\bar{n}_j^i \rightarrow \infty} \bar{n}_j^i P^i(n_j).$$

The $\Psi_j^i(x)$ functions can be determined from the corresponding limits for F_i :

$$\lim_{t \rightarrow \infty} F_i(t, e^{-\frac{s}{\bar{n}_1(t)}}, 1) = \lim_{\bar{n}_1 \rightarrow \infty} \sum_{n_1} P^i(n_1) e^{-\frac{s}{\bar{n}_1} n_1} = \int_0^\infty dx \Psi_1^i(x) e^{-sx} \\ = \hat{\Psi}_1^i(s), \quad (9)$$

$$\lim_{t \rightarrow \infty} F_i(t, 1, e^{-\frac{s}{\bar{n}_2(t)}}) = \lim_{\bar{n}_2 \rightarrow \infty} \sum_{n_2} P^i(n_2) e^{-\frac{s}{\bar{n}_2} n_2} = \int_0^\infty dx \Psi_2^i(x) e^{-sx} \\ = \hat{\Psi}_2^i(s).$$

Relations (9) show that the generating F_i functions in the indicated limits become the Laplace transformations $\hat{\Psi}_j^i(s)$ of the corresponding KNO functions $\Psi_j^i(x)$. This is an important result. Although the $P(n)$'s are hard to find, their asymptotic forms can be easily calculated for large \bar{n} , if the explicit form of the F generating functions is known. We find for the solution the following KNO functions of the quarks and gluons in the jets:

$$\Psi_1^1(x) = \Psi_2^1(x) = \frac{1}{\sqrt{2\pi x}} \exp(-x/2); \quad \Psi_1^2(x) = \Psi_2^2(x) = \exp(-x). \quad (10)$$

In conclusion, we shall examine two problems. First, how are the results for quarks and gluons related to the observed hadron distributions? There are reasons⁷ to believe that the production of hadrons (a process outside the scope of the perturbation theory in QCD) occurs "locally" within the quark-gluon clusters of finite mass $\sim Q_0$. In this case the final characteristics of hadrons are similar to the corresponding characteristics.

The second problem involves the correct conversion from the infrared divergences to the evolution equations for QCD. Thus, the naive derivation of Eqs. (4) in QCD gives infinite parameters λ and γ_2 (but a finite λ/γ_2 ratio) and some regularization procedure is required (see Ref. 6). For this reason, only those relations which give relative values [for example, Eqs. (8) and (10)] that depend only on λ/γ_2 can be trusted. The problem of calculating the absolute values [for example, Eq. (7)], which requires a detailed study of the emission mechanisms of soft partons, is now being investigated.

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¹A. M. Polyakov, Zh. Eksp. Teor. Fiz. **60**, 1572 (1971) [Sov. Phys. JETP **33**, 850 (1971)].

²S. Orfanidis and V. Rittenberg, Phys. Rev. D **10**, 2892 (1974).

³L. N. Lipatov, Yad. Fiz. **20**, 181 (1974) [Sov. J. Nucl. Phys. **20**, 94 (1975)]; G. Altarelli and G. Parisi, Nucl. Phys. B **126**, 298 (1977).

⁴Yu. L. Dokshitzer, D. I. D'yakonov, and S. I. Troyan, Materialy XIII Zimnei shkoly LIYaF (Proceedings of XIIIth LIYaF Winter School), Vol. 1, Leningrad, 1978, p. 3.

⁵B. A. Sevast'yanov, Vetyashchiesya protsessy (Branching Processes), Nauka, Moscow, 1971.

⁶K. Konishi, A. Ukawa, and G. Veneziano, Preprint RL-79-026, 1979.

⁷D. Amati and G. Veneziano, Phys. Lett. B **83**, 87 (1979).