

Self-focusing of waves on the surface of deep water

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Using a hyperbolic nonlinear Schrödinger equation of the kind that is often used to describe waves on the surface of deep water, we establish the existence of a new region of instability for plane waves of small, but finite amplitude. The unstable mode can propagate in a narrow solid angle about the normal to the direction of propagation of the main wave.

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The envelope of a small-amplitude wave on the surface of deep water satisfies the equation¹

$$i \frac{\partial \Phi}{\partial t} + \frac{\partial^2 \Phi}{\partial x^2} - 2 \frac{\partial^2 \Phi}{\partial y^2} - \Phi + |\Phi|^2 \Phi = 0. \quad (1)$$

In this letter we examine the stability of a plane-wave envelope Φ satisfying Eq. (1). Assuming that Φ is a real function of x , we obtain the first integral of Eq. (1)

$$\frac{1}{2} \Phi_0^2 = B + \frac{1}{2} \Phi_0^2 - \frac{1}{4} \Phi_0^4, \quad (2)$$

where B is a constant of integration. Assuming that the amplitude a is small, we write

$$\Phi_0 = 1 + a \cos(k_0 x) + O(a^2),$$

This is consistent with

$$B = -\frac{1}{4} + a^2 + O(a^4), \quad k_0^2 = 2 + O(a^2).$$

We shall show that allowance for the finite size of a leads to a new unstable mode that does not appear in the linear approximation in a .

Examining the variation of Φ about $\Phi_0 = 1$, $B = -\frac{1}{4}$ and setting $\Phi = 1 + \delta\Phi$,

$$\delta\Phi = \delta\Phi_1 e^{i(xk + \omega t)} + \delta\Phi_2 e^{i(kx - \omega^* t)},$$

we obtain the linear approximation to the dispersion relation:

$$\omega^2 = -2(k_x^2 - 2k_y^2) + (k_x^2 - 2k_y^2)^2. \quad (3)$$

We restrict discussion to the case of wave numbers $k \ll 1$, for which the waves are unstable in the first quadrant for

$$\theta < \tan^{-1} (1/\sqrt{2}) \approx 35^\circ$$

where $k = (k_x, k_y) = k(\cos\theta, \sin\theta)$. For larger k the instability region is narrower. Recent studies of a refined hyperbolic nonlinear Schrödinger equation confirm this result.² The result is also preserved when the complete system of equations for deep-water surface waves is expanded to second order in a .³ Here we propose to solve the problem first for any a and then pass to the limit of small a .

If it is assumed that ω and k are small, but of the same order, so that the k^4 term in Eq. (3) can be neglected, while B (and hence a) is arbitrary in the interval

$$-\frac{1}{2} \leq B \leq 0,$$

one can find the generalized version of dispersion relation (3). It is of the following form (see Ref. 4 in which an analogous problem is solved; the essential feature of the method is that we expand quantities in powers of k , but choose Φ_0 to be an exact solution of Eq. (2)):

$$\left(\frac{\omega}{k}\right)^4 + D(\theta) \left(\frac{\omega}{k}\right)^2 \cos^2 \theta + A(\theta) \cos^4 \theta = 0,$$

$$D(\theta) = a_1(\theta) + b_1(\theta) + a_2 b_2,$$

$$A(\theta) = a_1 b_1,$$

$$a_1 = [2 q^4 B_1 / B_2 H] [1 + 8 B_2^2 + \tan^2 \theta (1 - q^2) / 3 q^4],$$

$$b_1 = [8 (1 - q^2)^2 B_2 / B_1 H] [1 - 2 B_1^2 \tan^2 \theta / (1 - q^2)],$$

$$a_2 b_2 = -8 (1 - x)^2 (1 - q^2 - x)^2 (1 - q^2) (2 - q^2) / B_1 B_2 H^2,$$

$$q^2 = 2 \sqrt{1 + 4B} / \sqrt{1 + 4B} + 1 < 1,$$

$$H = (1 - q^2 - x^2) (2 - q^2),$$

$$B_1 = x, \quad B_2 = 1 - (2 - q^2) x / 2 (1 - q^2),$$

$$x = E(q) / K(q).$$

(E and K are complete elliptic integrals).

Sketching the polar diagram of ω/k , we obtain the analog of the C.M.A. diagram used in plasma physics.^{5,6} For $B \approx -0.25$ one obtains a new unstable mode (see Fig. 1), which is localized about $\pi/2$ (and $3\pi/2$). Analogous "nearly perpendicular" insta-

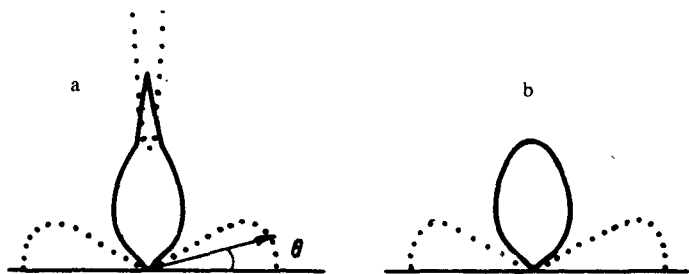


FIG. 1. Polar diagrams of $\omega/k(\theta)$: $a-B = -0.2495$, $a = 0.011$; $b-B = -0.25$, $a \rightarrow 0$. The solid curve is the true curve; the dotted curve is purely imaginary values of ω/k .

bilities are familiar in plasma physics.^{7,8} It seems probable that the usual expansion would give our new mode if terms of order a^4 were taken into account. This, however, would require much more effort than it took to obtain Eq. (4), which includes all powers of a^2 .

Preliminary investigation indicates that the model of Ref. 2, which is more refined than Eq. (1), should not alter our results substantially, since the essential terms are found in both models.

For all angles such that

$$(\cos \theta) > a$$

our polar diagram resembles the one obtained for $a \rightarrow 0$ (Fig. 1,b).

Thus we have shown that in the limit of small a a new instability is obtained in the dispersion relation

$$D(\omega, k, a) = 0$$

for deep-water waves described by a hyperbolic nonlinear Schrödinger equation. Our analysis is restricted to small k and therefore does not give the maximum increment; to find this would evidently require numerical calculations.

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