Distribution of resistivity probabilities of a finite, disordered system

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The average values of the resitivity ρ and conductivity σ are calculated for a finite, disordered system of length x. For $x \gg l$ (l is the free path length) it is found that $\langle \rho^n \rangle \sim \exp[(n^2+n)x/l]$ and $\langle \sigma^n \rangle \sim \exp(-x/4l)$. The resistivity distribution function is reconstructed and the resistivity logarithm is shown to be a Gaussian distribution.

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One of the chief difficulties in the theoretical study of disordered systems (DS) is the need to average physical quantities over a certain set of random parameters, for example, over the locations of impurities, over the exchange integrals, etc. In some cases such averaging can be performed explicitly, where the result obtained has a direct, physical meaning because of self-averagability of the quantity being studied (for example, the density of states and the thermodynamic characteristics associated with it). In the general case it is necessary to calculate the probability distribution of the values, rather than the average values. A disordered system can be characterized by some averaged parameter only if its distribution has a negligibly small width.

Arguments [1,2] have been presented in favor of the fact that the distribution $W(\rho,x)$ of the resistivities ρ for a disordered system of length x (the free path length is taken as a unit length) does not become narrow as $x \to \infty$. A calculation of $\langle \rho \rangle \sim \exp(\alpha x)$ and $\langle \rho^2 \rangle \sim \exp(\beta x)$ ($\alpha,\beta \sim 1$), which showed[2] that $\beta > 2\alpha$, was a direct demonstration of this assumption. This shows that the $W(\rho,x)$ distribution has a flat tail on the side of large ρ.

Our goal is to explicitly find $W(\rho,x)$ for a one-dimensional, disordered system with a disorder in the form of white noise, by using the method developed by Berezinskiĭ.[3] We shall show in particular that $\langle \rho^n \rangle \sim \exp[(n^2+n)x]$ and that the quantity $y = \ln \rho$ has a Gaussian distribution near $y_0 = x$ with a variance $(2x)^{1/2}$.

Let us define the dimensionless resistivity of the disordered system in terms of its transmission coefficient T by using the relationship $\rho = 1/T$. By introducing the reflection coefficient $R = 1 - T$, we can relate $\langle \rho^n \rangle$ to $\langle R^m \rangle \equiv R_m$.

$$\langle \rho^n(x) \rangle = \langle (1 - R(x))^{-n} \rangle = \sum_{m=0}^{\infty} \frac{(m+n-1)!}{m!\,(n-1)!} R_m(x) \approx \frac{1}{(n-1)!} \int_0^{\infty} m^{n-1} R_m(x)\,dm. \tag{1}$$

We have taken into account that $m \gg 1$ accounts for the major contribution to the sum when $x \gg 1$. The values $R_m(x)$ obey the equation[3]

$$\frac{dR_m}{dx} = m^2 (R_{m+1} + R_{m-1} - 2R_m); \quad R_m(0) = \delta_{mo}. \tag{2}$$

A Laplace transformation with respect to λ gives

$$\lambda R_m - \delta_{mo} = m^2 (R_{m+1} + R_{m-1} - 2R_m), \tag{3}$$

where λ is the Laplace parameter. For $m \gg 1$ this equation can be replaced by a differential equation whose solutions, as it is easy to see, have a power form with respect to m. After eliminating the solution that increases with m and determining the second integration constant from a comparison with the solution for $m \sim 1$ (see Ref. 4), we obtain

$$R_m(\lambda) = \frac{\pi^{1/2}}{2\lambda} (4m)^{-q} \frac{\Gamma(q+1)\Gamma(q+2)}{\Gamma(q+3/2)}; \quad q = \sqrt{\lambda + 1/4} - 1/2. \tag{4}$$

We should take into account for the inverse Laplace transformation that $R(\lambda)$ has a pole at $\lambda = 0$ and a branching at $\lambda = -\frac{1}{4}$. In the limit $\ln m > x \gg 1$ we obtain

$$R_m(x) = \frac{a}{\pi^{1/2}} \frac{\Gamma^3(a+1/2)}{(a-1/2)\,\Gamma(2a+1)} \left(\frac{m}{x}\right)^{1/2} \exp\left[-\left(a^2 + \frac{1}{4}\right)x\right]; \quad a \equiv \frac{\ln m}{2x}. \tag{5}$$

After substituting Eq. (5) in Eq. (1), we can see that the integral with respect to a has a saddle point near $a = n + \frac{1}{2}$, so that

$$\langle \rho^n \rangle = \frac{1}{2^n (2n-1)!!} \exp[(n^2 + n)x]. \tag{6}$$

It is easy to see that the probability distribution, which gives the moments (6), is

$$W(\rho, x) = \frac{aB(a+1/2,\, a+1/2)}{(\pi \rho x)^{1/2}} \exp\left[-\left(a^2 + \frac{1}{4}\right)x\right]; \quad a \equiv \frac{\ln \rho}{2x}, \tag{7}$$

where B is the Euler function. It follows from this that the quantity $y = \ln \rho$ has a Gaussian distribution

$$W(y, x) = (4\pi x)^{-1/2} \exp[-(y-x)^2/4x] \tag{8}$$

with a variance $\overline{\Delta y} = (2x)^{1/2} \ll \bar{y} = x$.

For comparison, we calculate the conductivity moments $\langle \sigma^n \rangle \equiv \langle T^n \rangle$. Because $T = Z_0(x)$ [the value $Z_m(x)$ was previously introduced by Berezinksiĭ; see also Ref. 4], our problem reduces to the calculation of $Z_0^{(n)}(x)\equiv\langle Z_0^n(x)\rangle$. Just as in Ref. 3, we can introduce

$$Z_m^{(n)} = \sum_{m_i} \left(\prod_{i=1}^{n} Z_{m_i}\right) \delta\left(\sum_{i=1}^{n} m_i - m\right). \tag{9}$$

These quantities obey the following system of equations:

$$\frac{dZ_m^{(n)}}{dx} = (m+n)^2 Z_{m+1}^{(n)} + m^2 Z_{m-1}^{(n)} - [(m+n)^2 + m^2 - n^2 + n] Z_m^{(n)},$$

$$Z_m^{(n)}(0) = \delta_{mo}. \tag{10}$$

We perform a Laplace transformation with respect to x and go over to a differential equation, taking into account the case $m \gg 1$. Thus, we obtain

$$\lambda Z_m^{(n)} - \delta_{mo} = (n^2 - n) Z_m^{(n)} + 2mn \frac{\partial Z_m^{(n)}}{\partial m} - m^2 \frac{\partial^2 Z_m^{(n)}}{\partial m^2}. \tag{11}$$

The solutions of this equation have the form

$$Z_m^{(n)}(\lambda) = f(\lambda) m^{-q}; \quad q = \frac{2n-1}{2} + \sqrt{\lambda + 1/4}. \tag{12}$$

The $f(\lambda)$ function can be found only by solving Eq. (11) for $m \sim 1$. It is important that Eq. (12), expressed as a function of λ, has a branch point at $\lambda = -\frac{1}{4}$. It is clear that this property holds for $Z_0^{(n)}(\lambda)$, so that after inverse Laplace transformation the main dependence of $Z_0^{(n)}$ on x has the form $\exp(-x/4)$. A more precise calculation using numerical methods makes it possible to find the pre-exponential factor, so that finally

$$\langle \sigma^n(x) \rangle \equiv Z_0^{(n)}(x) = \frac{\pi^{5/2}}{2} C(n)\, x^{-3/2} \exp(-x/4). \tag{13}$$

The $C(n)$ coefficients for n ranging from 1 to 5 have the values 1, 0.25, 0.14, 0.096, and 0.072. We can see that all the moments $\langle\sigma^n\rangle$ are equal within the accuracy of the numerical pre-exponential factor. This means that the main contribution in the averaging comes from the part of the distribution function of the transmission coefficient that has the form

$$w_1(T, x) = x^{-3/2} e^{-x/4} u(T), \tag{14}$$

where the $u(T)$ function is independent of x. The result obtained indicates that the probability of transit without scattering in a disordered system of length x decreases as $\exp(-x/4)$.

Generally, the probability distributions of the quantities $\sigma = T$ and $\rho = 1/T$ should be expressed in terms of each other in a trivial manner. Our calculation showed, however, that if the moments $\langle \rho^n \rangle$ and $\langle \sigma^n \rangle$ are known, then these functions can be reconstructed only in that region of variables which gives the main contribution to the corresponding moments. Of course, these regions turned out to be different for T and $1/T$.

Anderson *et al.*[1] obtained the probability distribution of the resistivity logarithm (8) and used it to calculate the average conductivity $\langle \sigma \rangle$, which was found to be independent of the length of the disordered system with exponential accuracy. Our results show that, because of their derivation method, Eqs. (7) and (8) can be used only for $\rho \equiv \ln y \gtrsim \exp(2x)$ [the saddle point, which determines the normalization of the function (7), is located at $\rho = \exp(2x)$]. In the calculation of the average of $\sigma = 1/\rho$ using (7), we can see that the integral converges at $\rho \sim 1$, i.e., outside the region of applicability of Eqs. (7) and (8). A direct calculation of $\langle \sigma^n(x) \rangle$ gives Eq. (13), which differs from that in Ref. 1 in the case $n = 1$.

A knowledge of the moments of the quantities $1/T$ and T makes it possible to reconstruct $W(1/T,x)$ in the region $1/T \gtrsim \exp(2x)$ [Eq. (7)] and $W(T,x)$ in the region $T \sim 1$ [Eq. (14)].

Let us now indicate the possible experimental evidence of the effects being considered. Experiments using isolated, one-dimensional disordered systems have not been performed heretofore. When such disordered systems form a quasi-one-dimensional conductor, their conductivities must be averaged, so that the net conductivity will be given by Eq. (13) for $n = 1$, which is multiplied by the number of disordered systems. Another possibility involves using a quantum-optical analogy. Let us assume that a plate of a transparent material has a random distribution of the refractive index along one direction. At a sufficient thickness this plate will almost completely reflect the light over a broad wavelength band, and the distribution of the transmission coefficient T for an ensemble of plates fabricated under statistically identical conditions will be defined by Eqs. (7) (after substituting $\rho = 1/T$) and (14). This means that an increase in the thickness of the plates will lead to a relative narrowing of the $\ln(1/T)$ distribution and to an exponential increase of the relative mean-square deviation of the value $1/T$, consisted with the fact that $\langle 1/T^2 \rangle \sim \langle 1/T \rangle^3$, as follows from Eq. (6) for $n = 1$ and $n = 2$.

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