Propagation of an amplifying pulse in a two-level medium

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Inverse-problem equations, which describe the propagation of an amplifying wave pulse in a medium with an inverse population, are constructed. A self-similar solution, which describes the propagation of such a pulse, is obtained and interpreted in terms of the inverse problem. A self-similar solution describes the compression of an amplifying pulse.

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1. The propagation of a wave pulse in a two-level medium, disregarding the dissipation effect, is described by the equations

$$E_t + E_x = 2i\rho; \qquad n_t = i(E\rho^* - E^*\rho); \qquad \rho_t = -2inE, \tag{1}$$

where E is the complex amplitude of a wave and ρ and n are the elements of the density matrix

$$\hat{\rho} = \begin{bmatrix} n & \rho \\ \rho^* & -n \end{bmatrix}.$$

It follows from Eqs. (1) that

$$n^2 + |\rho|^2 = A^2(x). \tag{2}$$

$A(x) > 0$ is the space density of the atoms that interact with the wave. In the steady state $\rho = 0$, and two situations are possible: $n = A(x)$, which corresponds to an inversely populated medium and $n = -A(x)$, which corresponds to a medium in the normal state. If $n \to -A(x)$ as $t \to \pm\infty$, then we are dealing with self-induced transparency.

We shall examine the case

$$\begin{matrix} n \to A(x) & & n \to -A(x) \\ t \to -\infty & & t \to \infty \end{matrix}, \tag{3}$$

which describes the transition of inversely populated atoms to the normal state. We analyze the problem on the semiaxis $0<x<\infty$; the incident pulse is given for $x=0$

$$E(x, t)\big|_{x=0} = E_0(t). \tag{4}$$

We obtain the following relation from Eqs. (1):

$$\int_{-\infty}^{\infty} |E|^2 dt = \int_{-\infty}^{\infty} |E_0|^2 dt + 2\phi(x) \tag{5}$$

which clearly shows that the pulse is amplified during propagation and absorbs energy that was previously contained in the inversely populated atoms.

This amplification can be described by a self-similar solution

$$\begin{matrix} E = \phi(x)\epsilon(\xi) & & n = A(x)N(\xi) \\ \rho = A(x)R(\xi) & & \xi = \phi(x)(t - x - t_0) \end{matrix} \tag{6}$$

where t_0 is an arbitrary constant. We have from Eqs. (1)

$$\begin{matrix} \xi\epsilon_\xi + \epsilon = 2iR & & N_\xi = i(\epsilon R^* - \epsilon^* R) \\ \int|\epsilon|^2 d\xi = 2 & & R_\xi = -2iN\epsilon \\ N(\infty) = -1 & & N^2 + |R|^2 = 1\,. \end{matrix} \tag{7}$$

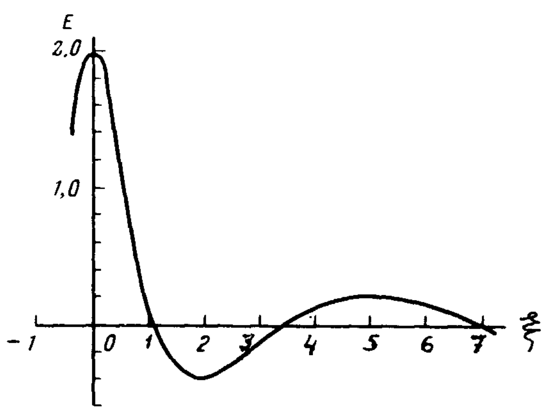

FIG. 1.

We can assume without loss of generality that the field ϵ is real and $R = -iW$ is purely imaginary. The general solution of Eqs. (7) has a singularity at $\xi = 0$. The physical solutions, which do not have a singularity, are characterized by the parameter $\epsilon_o = \epsilon|_{\xi=0}$, which has values in the limit $0 < \epsilon_o \leqslant 2$.

The solution is symmetrical $\epsilon(-\xi) = \epsilon(\xi)$ in the limiting case $\epsilon_0 = 2$, $W = 1$. The computer-generated plot of this solution is illustrated in Fig. 1. Equation (7) reduces to a type-3 Penleve equation in all the cases.

In a homogeneous medium when A = const the total energy flux through a given point is proportional to its distance from the origin and the pulse duration is inversely proportional to this distance. In the presence of a small attenuation $\partial E/\partial t \to (\partial/\partial t + \gamma)E$, a self-similar solution describes the initial phase $L \ll y/c$ of the formation of a stationary π pulse that propagates at the speed of light.

2. The method of the inverse-scattering problem (MISP) has already been applied[1,2] to the system (1). This application is based on the fact that the system (1) is a commutation rule $[L,A] = 0$ of two operators

$$\begin{aligned} L &= \partial_t - i(I\lambda + H) \\ A &= \partial x + i\left(\lambda I + H + \frac{\hat{\rho}}{\lambda}\right). \end{aligned} \qquad (8)$$

Here λ is the spectral parameter

$$I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \; H \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix}.$$

Until now, however, only the self-induced transparency has been analyzed. The boundary conditions (3) give rise to a number of essentially new effects from the point of view of the MISP theory.

We shall determine the Jost function for the ψ solution of the equation $L\psi = 0$, which has the asymptotic form

$$\psi \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\lambda t - i\left(\lambda x - \frac{\phi(x)}{\lambda}\right)} \qquad t \to +\infty$$

$$\psi \to \begin{pmatrix} a(\lambda, x)e^{i\lambda t} \\ b(\lambda, x)e^{-i\lambda t} \end{pmatrix} e^{-i\left(\lambda x - \frac{\phi(x)}{\lambda}\right)} \qquad t \to -\infty .$$

We determine from the condition $A\psi = 0$

$$a(\lambda, x) = a_o(\lambda)\, e^{\frac{-2i\phi(x)}{\lambda}}, \qquad b(\lambda, x) = b_o(\lambda)\, e^{2i\lambda x}.$$

Here $a(\lambda,x)$ and $b(\lambda,x)$ are the elements of the transition matrix (see Ref. 3), $a_o(\lambda)$ and $b_o(\lambda)$ are their values for $x = 0$, which are determined by the initial pulse $E_o(t)$, and $a(\lambda,x)$ is the analytic function in the upper half-plane of λ, which can have zeros at the points $\lambda = \lambda_n$, $Im\lambda_n > 0$. The $a(\lambda,x)$ function in the conventional scattering theory (see Ref. 3) is continuous on the λ real axis and the number of zeros in this case is finite. In our case $a(\lambda,x)$ has an essential singularity at the point $\lambda = 0$, which can be treated as an infinite-order zero. In this case $\int_{-\infty}^{\infty} |E| dt > \infty$, although $\int_{-\infty}^{\infty} |E|^2 dt < \infty$.

The inverse-scattering problem looks as follows in the presence of an essential singularity. The Gel'fand-Levitan-Marchenko equation has the form

$$K(t, y, x) = F^*(t + y, x) - \int_{-\infty}^{t} \int_{-\infty}^{t} K(t, s, x) F(s + s', x) F^*(s' + y, x)\, ds\, ds'$$

$$E(t, x) = 2iK(x, x, t). \tag{9}$$

The $F(t,x)$ function, which determines the solution, can be broken down into a sum of the terms $F = F_1 + F_2$. F_1 is expressed by the initial pulse

$$F_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b(\lambda, x)}{a(\lambda, x)} e^{-i\lambda t} d\lambda + \sum_n A_n e^{-i\lambda_n t + 2i\left(\lambda_n x + \frac{\phi(x)}{\lambda_n}\right)}. \tag{10}$$

Here A_n are constants which determine the initial location of solitons.

The function has the form

$$F_2 = \frac{1}{2\pi i} \int \frac{s(\lambda)}{a_0(\lambda)} e^{-i\lambda t + 2i\left(\lambda x + \frac{\phi(x)}{\lambda}\right)} d\lambda, \tag{11}$$

where $s(\lambda) \not\equiv 0$ is an arbitrary analytic function in the neighborhood of $\lambda = 0$, such that Eq. (9) is solvable. The integral in Eq. (11) is taken over a small neighborhood with the center at the point $\lambda = 0$.

3. The system (1) has solutions even if $E_0(t) \equiv 0$. These solutions are called spontaneous solutions. For the spontaneous solutions, $F_1 \equiv 0$. An infinite set of motion integrals is associated with the L operator

$$\frac{\partial R_n}{\partial x} = \frac{\partial Q_n}{\partial t}.$$

Here R_n is a polynominal of E, of E^*, and of their time derivatives; $R_1 = |E|^2$. We can show that

$$I_n = 0 \quad \text{for } n > 0 \qquad I_1 = 2\phi(x)$$

for all the spontaneous solutions. The self-similar solution is the simplest solution among the spontaneous solutions. For it

$$F_2 = \frac{s_0}{(2\pi)^2} \sqrt{\frac{2\phi(x)}{t - x}}\, J_1\left(2\sqrt{2\phi(x)(t - x)}\right).$$

Here J_1 is a Bessel function. The constant s_0 is related to the parameter E_0 in the solution of Eq. (6); this relation must be determined from the equation.

The existence of spontaneous solutions, which is attributed to the physical instability of the medium with an inverse population, indicates that the problem (1)–(4) is mathematically incorrect. The spontaneous values "grow" out of the small fluctuations that occur as $t \to -\infty$. The problem (1)–(4) can be corrected if we assume that these fluctuations are missing. It is simpler to complete the definition of the problem if $E_o(t) \equiv 0$ when $t < t_1$, where t_1 is a certain instant of time. The extension of the definition is the causality requirement (absence of velocity greater than that of light)

$$E(x, t) \equiv 0 \qquad \text{for } t < t_1 + x. \tag{12}$$

The $b(\lambda)$ coefficient now becomes an analytic function at $Im\lambda > 0$.

The causality requirement (12) allows us to determine $s(\lambda)$ uniquely. It turns out that $s(\lambda) = ib(\lambda)$. The F function in this case is defined by Eq. (10), where the integration is performed by bypassing the special point $\lambda = 0$ from above. Such effect was analyzed in Ref. 4.

We can show that the self-similar solution (6) is an asymptotic solution of the system (1), as $x \to \infty$, for almost any shape of the initial pulse.

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