Multiloop contribution to string theory

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The contribution made to the string amplitudes by surfaces of arbitrary topology is calculated. This calculation is equivalent to incorporating multiloop diagrams in string theory.

String theory is attracting increasing interest. It is thought likely that a theory of a fermion string in a ten-dimensional space (more precisely, a Green-Shvarts superstring theory, which is apparently equivalent to the theory of a fermion string) is a fundamental theory, capable of describing all existing interactions in the low-energy limit. In the present letter we examine the contribution of surfaces of arbitrary topology to the string amplitudes (the objective is to incorporate multiloop diagrams). We use Polyakov's formalism[1] of string theory, in which (as Polyakov pointed out later) a calculation of the g-loop contribution reduces to an evaluation of integrals over some finite-dimensional superspace (a superconformal space of moduli V_f). This space, however, has yet to be described, and the measure of an integration in it has yet to be constructed.

Our basic purpose here is to study the measure of integration which arises in this space and also the measure, in its boson analog (a conformal space of moduli V_b), which arises in an analysis of a boson string in critical dimensionality ($d = 26$). There is the hope that our results will be useful in analyzing the finiteness of string amplitudes, since the volume of the space of moduli, calculated in accordance with the natural metric in this space, turns out to be finite.

We understand "superconformal manifold" as a supermanifold spliced from superregions of complex dimensionality (1,1) by means of a superconformal transformations: transformations of the type $z \to u\,(z - \epsilon(z)\theta)$, $\theta \to \sqrt{u'(z)}\,[\theta + \epsilon(z) + \frac{1}{2}\epsilon(z)\epsilon'(z)\theta\,]$, where $u(z)$ and $\epsilon(z)$ are even and odd analytic functions of the even complex coordinate z, and θ is an odd complex coordinate. The underlying manifold M of the superconformal manifold is assumed here to be a compact surface of the type $g > 1$ (this assumption means that we are restricting the analysis to multiloop contributions to the theory of a closed string). The superconfomal space of moduli V_f is defined as the set of classes of superconformal manifolds with respect to superconformal equivalence. It turns out that any superconformal manifold of the type $g > 1$ is equivalent to a superconformal manifold which is found by factorization from the superanalog of the half-plane of a superregion $\mathscr{H}$ which coordinates $Z = (z,\theta)$, where z is an even complex coordinate that obeys the condition $\mathrm{Im} z > 0$, and θ is an odd complex coordinate. More precisely, this manifold is found from $\mathscr{H}$ by means of a factorization on the basis of the discrete subgroup Γ of the group $\mathscr{A}$ of real superprojection transformations: superconformal transformations γ for which we have $u(z) = (az + b)\,(cz + d)^{-1}$, $\epsilon(z) = \epsilon_1 + \epsilon_2 z$, where $a,b,c,d,\epsilon_1,\epsilon_2$

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are real, and $ad - bc = 1$. The transformations of the group Γ must satisfy the condition $|a+d| > 2$. The group Γ is isomorphic with the fundamental group $\pi_1(M)$, so that from it we can choose a system of generators $a_1, \dots, a_g, b_1, \dots, b_g$ which satisfy the relation $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$. Conversely, the specification of the system of elements $a_1, \dots, b_g$ determines the group Γ and thus the superconformal manifold $\mathscr{H}/\Gamma$. We may thus conclude that the dimensionality of the space V_f is $(6g-6, 4g-4)$ (it should be noted that the two conjugate subgroups Γ and Γ' of group $\mathscr{A}$ correspond to equivalent superconformal manifolds[2]). A "field of type k" on $\mathscr{M} = \mathscr{H}/\Gamma$ is a function $F(Z,\overline{Z})$ on $\mathscr{H}$ which satisfies the condition $F(\gamma Z, \gamma\overline{Z}) = [(1 + (1/2)\epsilon_1\epsilon_2)(cz+d) + \theta(\epsilon_2 d - \epsilon_1 c)]^{-k}(cz+d)^{2k} \cdot F(Z,\overline{Z})$ for $\gamma \in \Gamma$. In the set of fields of type k, we introduce a scalar product defined by $(F,G) = \int \overline{F} G Y^{-k} dV$, where $Y = \operatorname{Im}[z - (1/2)\theta\overline{\theta}], dV = Y^{-1} dz\, d\bar{z}\, d\theta\, d\overline{\theta}$, and the integral is over the fundamental region, which we assume to have a unit volume. The tangent space to V_f at the point defined by the supermanifold $\mathscr{M}$ can be identified with the space of odd analytical fields of type 3 on $\mathscr{M}$. The scalar product of these fields generates the metric on V_f; we denote the corresponding volume element by $d\nu_f$. In the space of even fields of type k we define the operator $\Box_k$ by $\Box_k = 2iYD\overline{D} - k(\theta - \overline{\theta})\overline{D}$, where $D = \partial/\partial\theta + \theta(\partial/\partial z)$. It can be shown that the measure that arises on V_f in a calculation of string amplitudes is given by

$$d\mu_f = |\det \Box_0|^{-5} |\det \Box_2| d\nu_f , \tag{1}$$

where det means the regularized determinant without allowance for zero modes.

Analogs of these results in the boson case are well known. In those results, the role of the group Γ is played by a subgroup of the group of projection transformations of the upper H half-plane. The measure $d\mu_b$ on V_b, which arises in a calculation of string amplitudes, is of the form

$$d\mu_b = (\det \Delta_0)^{-13} (\det \Delta_2) d\nu_b , \tag{2}$$

where $d\nu_b$ is the volume element corresponding to the so-called Weyl-Peterson metric on V_b, and Δ_k is the Laplacian on fields of weight k [expression (2) can be derived from the results of Ref. 3]. It can be shown that expression (2) can be put into the form

$$d\mu_b = \text{const}\, Z'(1)^{-13} Z(2)\, d\nu_b, \tag{3}$$

where $Z(s)$ is the Selberg zeta function corresponding to the group Γ. This function is defined by[4]

$$Z(s) = \prod_{[\gamma]} \prod_{k=1}^{\infty} (1 - \exp(-l_\gamma(s+k-1)), \tag{4}$$

where $\operatorname{tr}\gamma = |a+d| = 2\cosh l_\gamma/2$, and $[\gamma]$ runs over all the primitive classes of conjugate elements. An element $\gamma \in \Gamma$ is "primitive" if it cannot be represented in the form $\gamma = \beta^k$, where $\beta \in \Gamma$, $k \neq 1$. (The conjugation classes of primitive elements are in a mutually one-to-one correspondence with simple closed geodesics on H/Γ, and the numbers l_γ are equal to the lengths of these geodesics.) To find (4), we need to use Selberg's trace formula for forms of weight k (Ref. 4) and the arguments made in Ref. 5 for calculating the analytic twisting in the two-dimensional case. An expression for the volume element dv_b can be found from the results of Ref. 6. In length-twist coordinates in a Teichmüller space (a simply connected covering space of V_b) we have $dv_b = dl_1 \ldots dl_{3g-3} d\tau_1 \ldots d\tau_{3g-3}$. To describe the length-twist coordinates, we note that the surface of constant negative curvature of kind g can be cut along $3g-3$ closed geodesics into $2g-2$ surfaces with an edge, each of which is topologically equivalent to a circle with two holes. The length of these geodesics are explained by the coordinates $l_k, k = 1, \ldots, 3g-3$. If we fix some surface with coordinates $l_1, \ldots, l_{3g-3}$, we can find the other surfaces by cutting along each of $3g-3$ geodesics, rotating a distance τ_k along the geodesic of length l_k, and resplicing. The distances $\tau_1, \ldots, \tau_{3g-3}$ $(-\infty < \tau_k < \infty)$, along with the lengths $l_1, \ldots, l_{3g-3}$, form the coordinate system that we need.

After we had submitted this paper for publication, Yu. I. Manin, I. V. Frolov, and the present authors derived a supervariant of the Selberg trace formula. Using that formula, we can transform (1) to

$$d\mu_f = \text{const}\cdot Z(2)Z(3/2)^{-1}\left[Z'(1)\right]^{-5}Z(1/2)^{10}Z(0)^{-5}Z(-{}^{1}/_{2})^{-1}Z'(-1)dv_f,$$

where $Z(s)$ is constructed from (4), in which l_γ is determined by

$$2\cosh l_\gamma/2 = |a+d|\left(1+\frac{1}{2}\epsilon_1\epsilon_2\right) - 2\epsilon_1\epsilon_2.$$

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