

Comparison of the properties of liquid He³ and liquid He⁴

A. M. Dyugaev

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

(Submitted 17 October 1985)

Pis'ma Zh. Eksp. Teor. Fiz. **42**, No. 11, 442–444 (10 December 1985)

At high temperatures the law of corresponding states holds for liquid He³ and liquid He⁴: The difference between the masses of He³ and He⁴ atoms can be cancelled by choosing a temperature scale. This circumstance makes it possible to determine on the basis of the experimental data on He⁴ all thermodynamic and kinetic characteristics of He³ at $T > 4$ K.

1. In contrast with ideal gases, the exchange effects in liquid He³ and liquid He⁴ weaken with increasing density n . The degeneracy temperature, T_F , of He³ and the temperature at which He⁴ undergoes a transition to the superfluid state, T_λ , decrease with increasing n . There is therefore a region of n and T in which He³ and He⁴ are quantum liquids but not degenerate liquids^{1,2}: $T_{F,\lambda} < T < \bar{E}_{3,4}(n,0)$. In the entire temperature interval: $T_{F,\lambda} < T < E_\infty$, where E_∞ is the ionization energy of He³ and He⁴ atoms, the parameters $\bar{E}_3(n,0)$ and $\bar{E}_4(n,0)$ are the only energy scale dimensions of liquid He³ and liquid He⁴, since in this region of T it can be assumed that $T_F = T_\lambda = 0$ and $E_\infty = \infty$. Since the interaction potential of a He³ atom is the same as that of a He⁴ atom, the only difference between liquid He³ and liquid He⁴ at the indicated temperatures is the difference in their atomic masses. In this letter we show

that the parameters \bar{E}_3 and \bar{E}_4 differ by only the factor $\lambda = m_3/m_4 = 0.754 \cdot m_4 \bar{E}_4(n, 0) = m_3 \bar{E}_3(n, 0)$ if the densities are the same $n_3 = n_4 = n$. We also show that, having taken advantage of the freedom in choosing the temperature scale, this small difference in the parameters makes it possible to obtain a relationship between the kinetic and thermodynamic characteristics of liquid He³ and liquid He⁴:

$$\begin{aligned} C_3(n, T) &= C_4(n, \lambda T), & \eta_3(n, T) &= \eta_4(n, \lambda T), \\ \kappa_3(n, T) &= \frac{1}{\lambda} \kappa_4(n, \lambda T). \end{aligned} \quad (1)$$

Here C_3 and C_4 are the specific heats per particle, κ_3 and κ_4 are the thermal conductivities, and η_3 and η_4 are the viscosities of He³ and He⁴, respectively. The accuracy of relations (1), which can be used at $T > 4/3 T_\lambda$, increases with increasing n and T . For the temperature dependence of the pressures P_3 and P_4 we obtain the relation

$$P_3(n, T) = \frac{1}{\lambda} P_4(n, \lambda T) + n^2 \beta. \quad (2)$$

The parameter β is independent of n and T . Expression (2) can be used for all values of T , since the exchange contribution to the pressures P_3 and P_4 is small. If we assume that the He³ and He⁴ atoms are rigid spheres of radius r_0 , and if their statistical base is not differentiated, relations (1) and (2) will be exact and the parameter β will be zero. A system of rigid spheres actually does not have an energy scale other than the parameter \hbar^2/mr_0^2 , and relations (1) and (2) can be obtained by analyzing the dimensions of C , P , η , and κ in a standard manner.³ Accordingly, to justify relations (1) and (2), we need only to show that a rigid-sphere approximation is valid for liquid He³ and liquid He⁴ and to define the parameter β in (2).

2. We now turn to an exact expression for the total energy of a liquid per particle, $E^+ = \bar{E} + \bar{V}$.

$$\bar{E}(n, T) = \frac{\nu}{n} \int n_p(n, T) \frac{p^2}{2m} \frac{d^3 p}{(2\pi)^3}; \quad \bar{V} = \frac{n}{2} \int v(r) g(r, n, T) d^3 r. \quad (3)$$

Here n_p is the particle momentum distribution function, g is a pairing correlation function, v is a two-body potential, $\nu = 1$ for He⁴, and $\nu = 2$ for He³. Liquid He³ and liquid He⁴ differ from other liquids in that their short-range order is poorly defined. On the other hand, He³ and He⁴ differ markedly from ideal gases, since the occupation numbers n_p are small even at low T and the effects associated with the statistical base of the atoms are weakened. Ignoring the short-range order and the exchange effects, i.e., assuming that the liquids are completely structureless both in the coordinate space and in the momentum space, we can determine the average potential energy \bar{V} on the basis of a semiclassical estimate: $g_3 = g_4 = 1$ for $r > \tilde{r}_0$; $g_3 = g_4 = 0$ for $r < \tilde{r}_0$. The reversal point \tilde{r}_0 can be found from the relation $v(\tilde{r}_0) = \bar{E}$. Since at small r the potential v increases exponentially⁴

$$v(r) = v_0 \left\{ \exp \left[\gamma \left(1 - \frac{r}{r_0} \right) \right] - \left(\frac{r_0}{r} \right)^6 \right\} \quad \begin{array}{l} \gamma \approx 11 \\ v_0 \approx 40 \text{ K} \\ r_0 \approx 2.76 \text{ \AA} \end{array}$$

the value of \bar{r}_0 depends only slightly on T even in the classical limit when $\bar{E} = (3/2)T$:

$$\bar{r}_0 \approx r_0 \left(1 - \frac{1}{\gamma} \ln \frac{T}{v_0} \right) \quad T \gg v_0.$$

Accordingly, V depends only slightly on T and on the atomic mass m .

$$\bar{V} = \frac{1}{2} n \bar{v} \quad \bar{v} = \int_{r_0} v(r) d^3 r.$$

At $T=0$ the average kinetic energy as a function of m can be found from the identity

$$m \frac{\partial E^+(n, 0)}{\partial m} = -\bar{E}(n, 0). \quad (4)$$

Since the dependence of \bar{V} on m is ignored, it follows from (3) and (4) that $\bar{E}(n, 0) = \hbar^2/ml^2$, where l is the length parameter which depends on n but is independent of m .

The energy $\bar{E}(n, 0)$ is the only quantum scale of the liquid, since the parameter \bar{V} does not contain \hbar in the chosen approximation. Noting, therefore, that in the classical limit $\bar{E} = (3/2)T$ even for a nonideal liquid,⁵ we can determine the T dependence of \bar{E} ,

$$\bar{E}(n, T) = \bar{E}(n, 0) f(n, \tau) \quad \tau = \frac{T}{\bar{E}(n, 0)}. \quad (5)$$

At $T=0$ the value of \bar{E} thus determines the scale of \bar{E} for all values of T . The function f in (5) is normalized by the conditions $f(0) = 1$ and $f = (3/2)\tau$ for $\tau \gg 1$. These results do not depend on the explicit form of the function f . The only important point is that this function should be the same for both He³ and He⁴.

The expressions for C , P , and the parameter β can now be obtained on the basis of thermodynamic identities,

$$C = \partial E^+ / \partial T \quad P - T \frac{\partial P}{\partial T} = n^2 \frac{\partial E^+}{\partial n}.$$

The parameter β can be expressed in terms of the density of He³ at $T=0$ and $P=0$:

$$n^2 \beta = - \frac{1}{6} n^2 \bar{v} = \frac{n^2}{n_0^2} 11,72 \text{ bar}; \quad n_0 = 0.01635 \text{ \AA}^{-3}.$$

The value of n_0 corresponds to the molar volume $V_0 = 36.84 \text{ cm}^3/\text{mole}$.

3. The law of corresponding states for quantum liquids is described by relations (1) and (2). The physical meaning of this law is such that it is possible to find

combinations of observable quantities that do not have a Planck's constant. It is clear, therefore, that relations (1) and (2) have the correct classical limit when $\bar{E}(n,0) \ll T$. The relationship between C_3 and C_4 in (1) is consistent within 1% with the data of Ref. 6 in the ranges $7 < T < 29$ K and $2.3 < n/n_0 < 3$. The relation for κ_3 and κ_4 in (1) is in agreement with the data of Ref. 7 at $T \approx 4$ K. The relationship between P_3 and P_4 in (2) is consistent with the data of Ref. 8 at $T < 13$ K. There are surprisingly few data on He^3 in the region $T \approx 3\text{--}5$ K and $n \approx n_0$, where the exchange effects are particularly well defined. It is important to carry out special measurements of C , P , κ , and η on the isochores $n_3 = n_4$ at $4T_4 = 3T_3$ in order to determine the region in which the relations obtained by us can be used. Measurements of this sort would give a unique opportunity to distinguish the quantum effects from the exchange effects for liquid He^3 and liquid He^4 .

It is a pleasure to thank D. E. Khmel'nitskiĭ and G. M. Éliashberg for discussions and L. A. Tolochko for assistance with the experiment.

¹A. M. Dyugaev, Zh. Eksp. Teor. Fiz. **87**, 1232 (1984) [Sov. Phys. JETP **60**, 704 (1984)]; Phys. Lett. **108A**, 105 (1985).

²A. M. Dyugaev, Zh. Eksp. Teor. Fiz. **89**, 1220 (1985) [Sov. Phys. JETP **62**, No. 4 (1985)].

³J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids*, Wiley, New York (1954) (Russ. transl., IML, Moscow, 1961, p. 428).

⁴J. C. Slater and J. G. Kirkwood, Phys. Rev. **37**, 682 (1931).

⁵L. D. Landau and E. M. Lifshitz, *Statisticheskaya fizika (Statistical Physics)*, Nauka, Moscow, 1976, p. 116 [Pergamon Press, Oxford (1980), 2 vols., 3rd. ed.].

⁶J. S. Dugdale and J. P. Franck, Phil. Roy. Soc. (London) **257A**, 1 (1964).

⁷J. F. Kerrisk and W. E. Keller, Phys. Rev. **177**, 341 (1969).

⁸I. V. Bogoyavlenskii, L. V. Karnatsevich, and V. G. Konareva, Fiz. Nizk. Temp. **4**, 549 (1978) [Sov. J. Low Temp. Phys. **4**, 265 (1978)].

Translated by S. J. Amoretty