Exact solution of the $O(3)$ nonlinear two-dimensional σ-model

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An exact solution of the $O(3)$ nonlinear two-dimensional σ-model is given. The dependence of the ground-state energy on the external field, the spectrum, and the scattering amplitude is calculated by the Bethe method.

1. The $O(3)$ nonlinear σ-model (or the $\bar{n}$ field) describes the chiral field on the sphere S^2:

$$S = \frac{1}{2\lambda_0} \int d^2x \, (\partial_\mu \bar{n})^2 \, ; \qquad \bar{n} \in S^2 \, , \tag{1}$$

where $\bar{n} = (n_x, n_y, n_z)$ is a unit vector, and $\bar{n}^2 = 1$. This well-known model is possibly the simplest example of quantum-field theory in which the geometric properties of the manifold lead to a strong interaction of Goldstone particles, leaving them asymptotically free at short ranges,[1] and give the theory a nontrivial topology that allows the existence of instantons.[2] Because the interaction increases, the low-energy properties of the system, in particular, the particle spectrum, remain outside the scope of the standard methods of quantum-field theory. On the other hand, the $\bar{n}$ field is known to

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have an infinite set of conservation laws,[3,4] a property that enables the scattering to be factorized, and hence is fully integrable. Making use of just this property and the hypothesis that *the elementary particles are massive particles, that they belong to the isovector O(3) multiplet, and that they do not produce bound states,* Zamolodchikov and Zamolodchikov[5] determined the S matrix from the factorized bootstrap.

We will use here the Bethe-ansatz method to solve the σ-model completely. In particular, we will prove the hypothesis upon which the factorized bootstrap is based. For physical application, a study should be made of the behavior of the σ-model in an external uniform field:

$$\mathcal{E}(h) = -\ln \int D\bar{n}(x) \exp\{-S\{n(x)\} - \bar{h}_\mu \int d^2x \bar{J}_\mu\}, \tag{2}$$

where $h^2 = \bar{h}_\mu \bar{h}_\mu$, and $\bar{J}_\mu = \bar{n} x \partial_\mu \bar{n}$ is the Noether current. The ground-state energy is

$$\mathcal{E}(h) = \mathcal{E}(0) + m \int_{-F}^{F} \cosh\theta\, \epsilon(\theta) d\theta, \tag{3}$$

where $\epsilon(\theta)$ satisfies the equation

$$\epsilon(\theta) - \int_{-F}^{F} \frac{\epsilon(\theta')}{\pi^2 + (\theta - \theta')^2} d\theta' = h - m\cosh\theta; \quad \epsilon(\pm F) = 0. \tag{4}$$

2. This solution is based on an idea advanced by Polyakov[6] several years ago. Let us consider the main SU(2) chiral field with a broken right-hand symmetry

$$S = \int d^2x \left\{ \frac{1}{2\lambda_0}\left[(A_\mu^x)^2 + (A_\mu^y)^2\right] + \frac{1}{2\lambda_z}(A_\mu^z)^2\right\}; \tag{5}$$

$$A_\mu(x) = g^{-1}(x)\partial_\mu g(x) \equiv \bar{A}_\mu \bar{\sigma}; \quad g \in SU(2).$$

It is physically clear that if the moment of inertia $1/\lambda_z$ around the z axis is zero, the rotator becomes a vector on the sphere $S^2 = \mathrm{SU(2)/U(1)}$: $\bar{n} \to \hat{n} \equiv \bar{n}\bar{\sigma} = g\sigma^z g^{-1}$, and the action (1) will equal to the action (5). If $\lambda_z = \infty$, the action (5) has a U(1) gauge invariance with respect to the rotation of the rotators around the z axis: $g \to g \exp\left[\frac{i}{2}\phi(x)\sigma^z\right]$, which means that the field $\bar{n}(x)$ is determined from the U(1) subgroup of the adjacent SU(2).

The solution of the chiral isotropic ($\lambda_z = \lambda_0$) field has recently been found.[7] The method used in this solution can be immediately generalized to the anisotropic case. This method is based on the equivalence of the chiral field (5) and the integrable model (1 + 1) of the interacting fermions: Let us assume that $\psi \equiv \psi_f^\alpha$ ($\alpha = 1, 2,; f = 1 \ldots N_f$) is a fermion field that produces a "color" SU(2) multiplet and an auxiliary "flavor" U(N_f) multiplet and that $j_\mu^a = \bar{\psi}\gamma_\mu \sigma^a \psi$. A model with the Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial_\mu \gamma_\mu \psi + \sum_{a = x, y, z} \lambda_a j_\mu^a j_\mu^a \tag{6}$$

will then be equivalent to (5) in the limit $N_f \to \infty$ if $\lambda_x = \lambda_y = \lambda_0$.

The fermion model (6) was solved by the Bethe method for finite λ_z and N_f and then the limits $N_f \to \infty$ and $\lambda_z \to \infty$ were determined. The first limit, which is a regular limit in a certain sense, was discussed in Ref. 7. The main part of the solution, which is rather subtle, is the limit $\lambda_z \to \infty$. The reason for this is that in the case $\lambda_z = \infty$ we are dealing with a zero-current theory: $j^z(x) = 0$. This is the gauge-invariant theory. We will show that if λ_z is finite, i.e., if gauge invariance is broken, the elementary particles must be the SU(2) doublets. In the case $\lambda_z = \infty$, however, these particles undergo a transformation as a result of gauge transformation. This means that the particles are affected by the U(1) Coulomb potential, which increases with distance and which leads to the Schwinger confinement mechanism: The mass of the SU(2) doublets becomes infinite in the limit $\lambda_z \to \infty$ and they vanish from the physical spectrum. However, the bound states of these doublets, the $O(3)$ triplets, have a finite mass and are the elementary particles of the $\bar{n}$ field.

3. Omitting the particular details of the solution of the fermion model (6), we write the Bethe equations: the eigenstate of the $2\mathcal{N}$ particles, which is a flavor singlet, is described by the rapidities $\{k_1^{\pm} \ldots k_{\mathcal{N}}^{\pm}\}$ and $\{\Lambda_1 \ldots \Lambda_M\}$ that satisfy the equations

$$\exp(ik_j^{\pm}L) = \prod_{\alpha=1}^{M} e_{N_f}(\Lambda_\alpha \mp f\,|\,\mu); \tag{7a}$$

$$\prod_{(\pm)} [e_{N_f}(\Lambda_\alpha \pm f\,|\,\mu)]^{\mathcal{N}} = -\prod_{\beta=1}^{M} e_2(\Lambda_\alpha - \Lambda_\beta\,|\,\mu), \tag{7b}$$

where $e_n(x|\mu) = \sinh \mu(in/2 + x)/\sinh \mu(-in/2 + x), M = \mathcal{N}N_f - S^z$, and S^z — SU(2) is the isospin. The parameter $\mu \in [0, \pi/N_f]$ and f are nonuniversal functions of λ_0 and λ_z. We will take advantage of the following properties of these functions:$\mu(\lambda_z, \lambda_0)$ is a monotonic function of $\lambda_z - \lambda_0$ when $\lambda_z = \lambda_0$ and $\mu \to 0$; $\mu f \to 1/\lambda_0$ [Eqs. (7) are identical in this case to the equations for a chiral isotropic field[7]], and $\mu \to \pi/N_f$ as $\lambda_z \to \infty$.

4. The ground state of the system is formed by the spin complexes ("strings") of order N_f: $\Lambda^{(k)} = \Lambda + ik\,[k = -(N_f - 1) - (N_f - 1)]$. We asume that θ_α $(\alpha = 1 \ldots m)$ are the rapidities of the holes in a sea of N_f strings and that ρ_ν and λ_κ are rapidities that parametrize the strings of order larger and smaller than N_f. These rapidities are described by the so-called physical Bethe equations which we will write here without showing how they were derived

$$\exp(iM \sinh \theta_\alpha L)$$

$$= \prod_{\beta=1}^{m} S\left(\theta_\alpha - \theta_\beta'\,|\,\frac{\gamma}{8\pi}\right) S(\theta_\alpha - \theta_\beta\,|N_f) \prod_\nu e_1\left(\rho_\nu - \theta_\alpha\,|\,\frac{8\pi}{\gamma}\right) \prod_\kappa e_1\left(\lambda_\kappa - \theta_\alpha\,|\,\frac{1}{N_f}\right), \tag{8a}$$

$$\prod_{\alpha=1}^{m} e_1\left(\rho_\nu - \theta_\alpha\,|\,\frac{8\pi}{\gamma}\right) = -\prod_{\nu'} e_2\left(\rho_\nu - \rho_{\nu'}\,|\,\frac{8\pi}{\gamma}\right); \tag{8b}$$

$$\prod_{\alpha=1}^{m} e_1\left(\lambda_\kappa - \theta_\alpha \,|\, \frac{1}{N_f}\right) = - \prod_{\kappa'} e_2\left(\lambda_\kappa - \lambda_{\kappa'} \,|\, \frac{1}{N_f}\right) \cdot \tag{8c}$$

Here $\gamma = \left(\frac{\pi}{\mu} - N_f\right) 8\pi$, $M \propto \mathcal{N}/L \exp(-\pi f/2\mu)$ is the mass of the elementary particles, and

$$S(\theta \mid \gamma) = \exp\left\{ i \int_0^\infty \frac{d\omega}{\pi} \frac{\sin\frac{2}{\pi}\omega\theta}{\omega} \frac{\sinh\omega(\gamma/8\pi - 1)}{2\cosh\omega\sinh\omega\,\gamma/8\pi} \right\}. \tag{9}$$

Factorization of the physical Bethe equations into two parts that depend solely on γ and ρ and on N_f and λ means that the S matrix of the physical particles, $\hat{S}$, is a tensor product of two factorized S matrices. It follows from Eq. (8) that

$$\hat{S}\left(\theta \,|\, \frac{\gamma}{8\pi}\right) = \lim_{N_f\to\infty} \hat{S}\left(\theta \,|\, \frac{\gamma}{8\pi}\, N_f\right) = \lim_{N_f\to\infty} \hat{S}^{U(1)}\left(\theta \,|\, \frac{\gamma}{8\pi}\right) \otimes \hat{S}^{U(1)}(\theta \,|\, N_f) = \tag{10}$$

$$\hat{S}^{U(1)}\left(\theta \,|\, \frac{\gamma}{8\pi}\right) \otimes \hat{S}^{SU(2)}(\theta),$$

where $\hat{S}^{U(1)}(\theta \,|\gamma)$ and $\hat{S}^{SU(2)}(\theta)$ are the minimum, factorized, crossing-invariant S matrices that have U(1) and SU(2) invariances, respectively. These matrices are given in explicit form in Refs. 8 and 9 (see also the review by Kulish and Sklyanin[10]). This result has a simple physical meaning which reflects the global $U(1)_{\rm right} \otimes SU(2)_{\rm left}$ symmetry of model (5).

5. Let us now consider the limit $\lambda_z \to \infty$, i.e., $\gamma \to 0$. The properties of $\hat{S}^{U(1)}(\theta/\gamma)$ and $\hat{S}^{SU(2)}(\theta)$ are well known: $S^{SU(2)}(\theta)$ does not have any singularities on the physical sheet; $\hat{S}^{U(1)}(\theta\,|\gamma)$ is the S matrix of the sine-Gordon model,[8] which at $\gamma < 8\pi$ has poles $\theta = i\pi - ik^\gamma/8$, giving rise to the spectrum of bound states with masses: $m_k = 2M \sin^{k\gamma}/16$ ($k = 1 \ldots < 8\pi/\gamma$). The mass of the SU(2) doublets, M, becomes infinite in the limit $\gamma \to 0$: $M \propto \gamma^{-1}$. In contrast, the masses of bound states remain finite: $m \equiv m_1 \propto M\gamma$, whereas the higher bound states decay into "elementary" states ($k = 1$): $m_k \to k m_1$. An elementary particle (a lower bound state) is a reducible SU(2) $\otimes$ SU(2) tensor—a compound state comprised of an $O(3)$ triplet ($\bar{\pi}$) and a singlet (η). The scattering amplitudes of these particles can be determined by a standard procedure of "gluing" (Refs. 11 and 12).[1)] It can be shown that as $\gamma \to 0$, the triplet states split off from the singlet states: $\langle \eta\eta | \bar{\pi}\bar{\pi} \rangle \propto \gamma \to 0$; $\langle \eta\bar{\pi} | \eta\bar{\pi} \rangle \propto \gamma^{1/2} \to 0$, and the S matrices $S_{\bar{\pi}\bar{\pi}} \equiv \langle \pi^i\pi^k | \pi^j\pi^l \rangle$ and $\langle \eta\eta | \eta\eta \rangle$ become unitary matrices. Here $S_{\bar{\pi}\bar{\pi}}$ is the $O(3)$ S matrix that was determined in Ref. 5 (the uncoupled subspace of the singlet particles is an artifact associated with the gauge-group orbit).

6. The phenomenon we have described manifests itself in the Bethe method in the following way. Let us consider, for simplicity, the case in which $8\pi/\gamma$ is an integer. The rapidities ρ_ν and the second term in Eq. (8a) will then vanish. The factor $S(\theta\,|\gamma/8\pi)$ has a pole on the physical sheet. This means that Eqs. (8) have complex solutions;

$\theta = \Theta^{(\nu)} \pm i[(\pi/2) - (\gamma/16)k]$; $k = 1 \ldots \nu$, which describe the bound states. The equations for the rapidities $\Theta^{(\nu)}$ are derived from Eqs. (8). For elementary particles, for example, we have ($\Theta^{(1)} \equiv \Theta$)

$$\exp(im \sinh \Theta L) = \prod_{\theta} S^{(1)}(\Theta - \theta) \prod_{\Theta^{(\nu)}} S^{(1,\nu)}(\Theta - \Theta^{(\nu)}) \times \prod_{\lambda} \frac{\left(i\pi - \frac{i\gamma}{16} - \Theta + \lambda\right)\left(+\frac{i\gamma}{16} - \Theta + \lambda\right)}{\left(-i\pi + \frac{i\gamma}{16} - \Theta + \lambda\right)\left(-\frac{i\gamma}{16} - \Theta + \lambda\right)};$$

$$\prod_{\theta} \frac{\frac{i\pi}{2} + \lambda - \theta}{\frac{-i\pi}{2} + \lambda - \theta} \prod_{\Theta^{(\nu)}} \prod_{k=1}^{\nu} \prod_{(\pm)} \frac{\frac{i\pi}{2} \mp i\left(\frac{\pi}{2} - \frac{\gamma}{16}k\right) - \Theta^{(\nu)} + \lambda}{\frac{-i\pi}{2} \mp i\left(\frac{\pi}{2} - \frac{\gamma}{16}k\right) - \Theta^{(\nu)} + \lambda} \tag{11}$$

$$= - \prod_{\lambda'} \frac{i\pi + \lambda - \lambda'}{-i\pi + \lambda - \lambda'}; \tag{12}$$

where

$$S^{(1)}(\theta) = \tilde{S}\left(\theta + \frac{i\gamma}{16} - \frac{i\pi}{2}\right)\tilde{S}\left(\theta - \frac{i\gamma}{16} + \frac{i\pi}{2}\right),$$

$$S^{(m,k)}(\theta) = \tilde{S}\left(\theta - \frac{i\gamma}{16}(m-k)\right)\tilde{S}\left(\theta + \frac{i\gamma}{16}(m-k)\right) \times \tilde{S}\left(\theta - \frac{i\gamma}{16}(m+k) + i\pi\right)\tilde{S}\left(\theta + \frac{i\gamma}{16}(m+k) - i\pi\right),$$

$$\tilde{S}(\theta) = S(\theta \mid \gamma/8\pi) S(\theta \mid \infty) \tag{13}$$

are the amplitudes for the scattering of elementary particles by solitons and other bound states. At $\gamma = 0$, the physical states consist solely of the elementary particles, whose rapidities satisfy, according to (11) and (13), the Bethe equations

$$\exp(im \sinh \Theta_\alpha L) = \prod_{\beta=1}^{p} \frac{\Theta_\alpha - \Theta_\beta - i\pi}{\Theta_\alpha - \Theta_\beta + i\pi} \prod_{\kappa=1}^{q} \frac{\Theta_\alpha - \lambda_\kappa + i\pi}{\Theta_\alpha - \lambda_\kappa - i\pi}; \quad (\alpha = 1 \ldots p)$$

$$\prod_{\alpha=1}^{p} \frac{\Theta_\alpha - \lambda_\kappa + i\pi}{\Theta_\alpha - \lambda_\kappa - i\pi} = - \prod_{\kappa'=1}^{q} \frac{\lambda_\kappa - \lambda_{\kappa'} + i\pi}{\lambda_\kappa - \lambda_{\kappa'} - i\pi}; \quad (\kappa = 1 \ldots q), \tag{14}$$

which solve the $O(3)$ nonlinear σ-model.

7. Equations (14) can be expressed as a problem involving eigenvalues:

$$\exp(imL \sinh \Theta_\alpha) = \prod_{\beta=\alpha+1}^{p} \hat{S}_{\bar{\pi}_\alpha \bar{\pi}_\beta}(\Theta_\alpha - \Theta_\beta) \prod_{\beta=1}^{\alpha-1} \hat{S}_{\bar{\pi}_\alpha \bar{\pi}_\beta}(\Theta_\alpha - \Theta_\beta), \tag{15}$$

where

$$(S_{\bar{\pi}\,\bar{\pi}}(\Theta))^{ij}_{kl} = (2i\pi\Theta\delta_{ij}\delta_{kl} + \Theta\delta_{ik}\delta_{jl} + 2\pi i\delta_{il}\delta_{jk})\frac{1}{(\Theta + i\pi)(\Theta - 2\pi i)}\,. \tag{16}$$

The operator on the right side of (15), which accounts for Eqs. (14), was diagonalized in Ref. 13 in another context. According to the general factorized scattering theory, $\hat{S}(\theta)$ is the S matrix of the physical particles. This matrix describes the minimum factorized scattering theory of the $O(3)$ isovector particle.[5]

8. The quantities p and q in Eqs. (14) describe the square of the Noether current and its projection $\bar{J}_0 = \int dx \bar{n} x \partial_0 \bar{n}$; $\bar{J}_0^2 = p(p+1)$; $J_0^z = p - 1$. Using the standard procedure (see Ref. 14, for example), we find that the energy in a uniform field h is given by Eqs. (3) and (4).

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[1)]The possibility of interpreting the $O(3)$ isovector particle as a bound state of the SU(2) doublets with an infinite mass in factorized scattering theory was discussed for the first time in Refs. 11 and 12.

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