Two-dimensional self-similar plasma streams with an azimuthal magnetic field

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Two-dimensional self-similar solutions of the magnetogasdynamic equations for the radial expansion of a plasma with an azimuthal magnetic field are studied. An explicit analytic expression for these solutions is found for the adiabatic index $\gamma = 4/3$.

Self-similar solutions are important in gasdynamics and various problems in the physics of plasmas, both astrophysical and laboratory. Most of the previous studies have dealt with problems of high symmetry, which essentially reduce to one-dimensional or quasi-one-dimensional problems (see Ref. 1 for one of the few exceptional cases). In the present letter we examine a class of problems of ideal MHD for a purely azimuthal orientation of the magnetic field. An example of such a problem is the expansion of a plasma with a spontaneous azimuthal magnetic field away from the focus of a laser beam. Various mechanisms for the production of such fields were studied in Refs. 2–5. In this case we obviously have only an azimuthal symmetry, and the plasma flow is definitely two-dimensional. We formulate a general self-similar problem of the expansion with an azimuthal magnetic field. We show that for the special case in which the adiabatic index of the plasma is $\gamma = 4/3$ the solution of the problem can be written in an explicit analytic form.

The equations of ideal magnetogasdynamics describing the flow of a plasma with an adiabatic index γ can be written

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T + (\gamma - 1) T \nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\nabla \times \mathbf{b}) = 0, \tag{3}$$

$$n\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) + \nabla (n T) + \frac{1}{\beta} \mathbf{b} \times (\nabla \times \mathbf{b}) = 0. \tag{4}$$

Here $\beta = \mu_0 p_0 / b_0^2$, and we are using the standard notation for the dimensionless variables T/T_0, n/n_0, b/b_0, and p/p_0.

We first consider whether it is possible to express the solution of this problem in the self-similar form $n = N(\xi)/a^k$, $\mathbf{b} = \mathbf{B}(\xi)/a^l$, $T = \Theta(\xi) a^m$, $\mathbf{v} = \dot{a}\mathbf{V}(\xi)$, where $a = a(t)$ is a scale value with the dimensionality of length, which depends on only the time, and $\xi = r/a$ is a self-similar coordinate. The time variable t can be separated in Eqs. (1)–(4) under the conditions (a) $2l = k + m$ and (b) $\dot{a}^2 \sim a^{-m}$. It follows from the latter condition that $a(t) \sim t^{2/m+2}$. Two of the parameters k, l, m can be chosen arbitrarily.

We transform to spherical coordinates (r, θ, φ) in (1)–(4), taking into account the azimuthal symmetry of the problems $(\partial/\partial\varphi = 0)$ and the fact that the magnetic field $B(\xi, \theta)$ has only an azimuthal component. We denote the radial and θ components of the velocity by $V(\theta, \xi)$ and $U(\theta, \xi)$, respectively, and we find from (1)–(4)

$$kN + \xi \frac{\partial N}{\partial \xi} - \frac{1}{\xi^2} \frac{\partial}{\partial \xi}(\xi^2 N V) - \frac{1}{\xi \sin\theta} \frac{\partial}{\partial \theta}(N U \sin\theta) = 0, \tag{5}$$

$$m\Theta + (\xi - V)\frac{\partial \Theta}{\partial \xi} - \frac{U}{\xi}\frac{\partial \Theta}{\partial \theta} - (\gamma - 1)\frac{\Theta}{\xi^2}\frac{\partial}{\partial \xi}(\xi^2 V) - (\gamma - 1)\frac{\Theta}{\xi \sin\theta}\frac{\partial}{\partial \theta}(U \sin\theta) = 0, \tag{6}$$

$$lB + \xi \frac{\partial B}{\partial \xi} - \frac{1}{\xi}\frac{\partial}{\partial \xi}(\xi B V) - \frac{1}{\xi}\frac{\partial}{\partial \theta}(B U) = 0, \tag{7}$$

$$N\left(\frac{\partial V}{\partial \xi}(V - \xi) - \Gamma V + \frac{U}{\xi}\frac{\partial V}{\partial \theta} - \frac{U^2}{\xi}\right) + \frac{\partial}{\partial \xi}(N\Theta) + \frac{1}{\beta}\frac{B}{\xi}\frac{\partial}{\partial \xi}(B\xi) = 0, \tag{8}$$

$$N\left(\frac{\partial U}{\partial \xi}(V-\xi)-\Gamma U+\frac{U}{\xi}\frac{\partial U}{\partial \theta}+\frac{UV}{\xi}\right)+\frac{1}{\xi}\frac{\partial}{\partial \theta}(N\Theta)+\frac{1}{\beta}\frac{B}{\xi\sin\theta}\frac{\partial}{\partial\theta}(B\sin\theta)=0. \tag{9}$$

where $\Gamma = +a\ddot{a}/a^2 = -m/2$. The boundary conditions are determined by the physics of the process. In general, a problem of this type would have to be solved numerically. In the present letter we point out certain cases in which this system of equations can be simplified substantially. For example, if $l = k-1$ and $B = N\xi\sin\theta$, Eqs. (5) and (7) become identical. The number of adjustable parameters then reduces to one: $m = k-2$, $a \sim t^{2/k}$. Solutions of this type for a plane geometry were studied in Ref. 6. Another simplification results in the case of an adiabatic expansion with a uniform entropy distribution: Equation (6) is satisfied with $\Theta \sim N^{\gamma-1}$ and $m = k(\gamma-1)$. Another important particular case is the isothermal expansion of a plasma ($\gamma = 1$). Without going into detail, we note that in this case it is possible to single out the angular dependence and to reduce the problem to ordinary differential equations.

Finally, we consider solutions of Eqs. (5)–(9) describing a purely radial expansion of a plasma ($U = 0$), in which the radial-velocity component depends on only ξ. The simplest solution of this type corresponds to a uniform deformation,

$$V(\xi) = \xi .$$

In this case, Eqs. (5)–(7) are satisfied identically with $k = 3$, $l = 2$, and $\gamma = 4/3$. Condition (b) for the time dependence is replaced here by the equation $\ddot{a} = \mu a^{-2}$, where $\mu = \text{const}$. Denoting $P = N\Theta$, and setting $\beta = 1$, we find from (8)–(9)

$$\frac{\partial P}{\partial \xi}+\frac{B}{\xi}\frac{\partial}{\partial \xi}(\xi B) = -\mu\xi N, \tag{10}$$

$$\frac{\partial P}{\partial \theta}+\frac{B}{\sin\theta}\frac{\partial}{\partial\theta}(B\sin\theta) = 0. \tag{11}$$

For the time dependence we find

$$t = \pm\left[a^{1/2}(a-s)^{1/2}+s\ln\left(a^{1/2}+(s-a)^{1/2}\right)\right], \tag{12}$$

where $s = \operatorname{sgn}\mu$, and positive and negative values of t correspond to expansion and compression, respectively. If $\mu < 0$, we also have solutions with a maximum final radius,

$$t = \pm\left[\arcsin a^{1/2} - a^{1/2}(1-a)^{1/2}\right],$$

and a power-law solution $a \sim |t|^{2/3}$. It can be shown that if $\mu = 0$, there are no nontrivial solutions of (10) and (11).

For an arbitrary function $B(\xi,\theta)$, we find from (11)

$$P(\xi,\theta) = P_1(\xi) - \int^{\theta}\left[\frac{B(\xi,\theta')}{\sin\theta'}\frac{\partial}{\partial\theta'}B(\xi,\theta')\sin\theta'\right]d\theta' \tag{13}$$

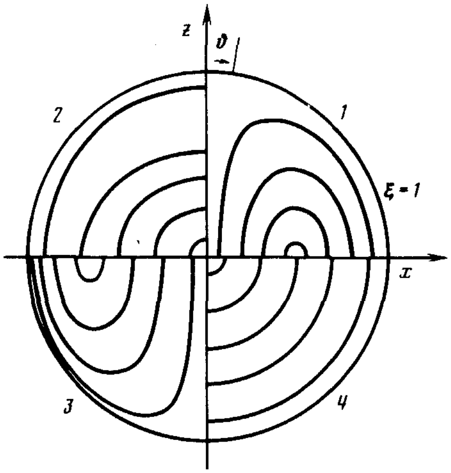

FIG. 1.

and $N(\xi, \theta)$ is expressed in terms of B and P with the help of (10). The solution is thus given by Eqs. (10), (12), and (13) with arbitrary functions $B(\xi, \theta)$ and $P_1(\xi)$ determined by the boundary conditions. The functions $P(\xi, \theta)$ and $N(\xi, \theta)$ should, of course, be nonnegative everywhere.

As an example we write the explicit simple solution describing the radial expansion of a plasma with an azimuthal magnetic field for the case in which the magnetic field and the plasma pressure vanish at the spherical boundary of the plasma. In this case we have $\mu > 0$, and the time evolution is determined by (12) with $s = 1$.

The spatial dependence is

$$B = \xi(1-\xi)\sin\theta,$$

$$N = 6 + (1-\xi)[(8-\sin^2\theta)\xi - 4],$$

$$P = 3(1-\xi^2) + \xi^2(1-\xi)^2(2-\sin^2\theta).$$

Figure 1 shows lines of $B =$ const, $N =$ const, streamlines, and lines of $P =$ const in sectors 1, 2, 3, 4, respectively.

If the magnetic field does not vanish at the plasma boundary, the solution must be joined with the corresponding solution of the linear wave equation for B in the vacuum region. In this case the lines of the conduction current that intersect the plasma boundary continue as lines of displacement current. The kinetic pressure may also remain finite at the plasma boundary if this boundary is a current sheet (a tangential discontinuity).

We note in conclusion that for a uniform deformation Eq. (3) is satisfied by an arbitrary function of the type $\mathbf{b} = \mathbf{B}(\xi)/a^2(t)$ with $\nabla\cdot\mathbf{B}' = 0$. Accordingly, for the case $\gamma = 4/3$ we can find self-similar solutions for more complex magnetic-field configurations. In particular, there exist families of such solutions for expansion (or compression) at a constant velocity of arbitrary equilibrium plasma configurations describable by solutions of the Grad-Shafranov-Schlüter equations: tokamaks, etc.

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[1]A. R. Garlick, Astrophys. Space Sci. **84**, 205 (1982).
[2]G. A. Askar'yan, M. S. Rabinovich, A. D. Smirnova, and V. B. Studenov, Pis'ma Zh. Eksp. Teor. Fiz. **5**, 116 (1967) [JETP Lett. **5**, 93 (1967)].
[3]V. V. Korobkin and R. V. Smirnov, Pis'ma Zh. Eksp. Teor. Fiz. **4**, 103 (1966) [JETP Lett. **4**, 70 (1966)].
[4]V. Sh. Abdullaev, Yu. M. Aliev, and V. Yu. Bychenkov, Pis'ma Zh. Eksp. Teor. Fiz. **28**, 524 (1978) [JETP Lett. **28**, 485 (1978)].
[5]C. E. Max, W. M. Manheimer, and J. J. Thompson, Phys. Fluids **21**, 128 (1978).
[6]D. Anderson, M. Bonnedal, and M. Lisak, Phys. Scripta **22**, 507 (1980).

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