Semiclassical regime of a three-dimensional wave collapse

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A nonlinear three-dimensional Schrödinger equation is used to construct a collapsing-type semiclassical solution which describes trapping of a finite number of waves at a singularity. The existence of this particular regime is confirmed by numerical simulations.

One of the basic models in the physics of nonlinear waves is the nonlinear Schrödinger equation

$$i\psi_t + \frac{1}{2}\Delta\psi + |\psi|^2\psi = 0, \tag{1}$$

which describes, in particular, the behavior of a spectrally narrow wave packet in a medium with a positive dispersion ($\omega_k'' > 0$) and inertialess nonlinearity. The solution of Eq. (1) depends essentially on the dimensionality of space, d. If $d \geqslant 2$, Eq. (1) describes a fundamental effect—a collapse of waves, wherein the amplitude space of a wave packet, ψ, becomes infinite at particular points. From the physical viewpoint, wave collapse, a process requiring a finite time, is a spontaneous concentration of wave energy in small regions of space, followed by its dissipation. If $d = 2$, a wave collapse can be interpreted as the formation of point foci due to self-focusing.[1] A negative value of the Hamiltonian of Eq. (1), $H = 1/2\int(|\nabla\psi|^2 - |\psi|^4)d\mathbf{r}$, at $t < 0$ is clearly a sufficient condition for a collapse to occur.[2,3] In some cases (problems involving Langmuir waves in a plasma, for example), a wave collapse may be the basic mechanism for wave-energy dissipation.

To determine the efficiency of this mechanism, we must know the amount of wave energy concentrated in the collapse zone. The question concerning the behavior of the amplitude ψ near the collapse zone therefore has a basic physical meaning. In the present letter we consider the $d = 3$ case, which is important from the viewpoint of plasma physics.

Equation (1) is consistent with a self-similar substitution (α is an arbitrary value), irrespective of the value of d (Ref. 3):

$$\psi = \frac{1}{(t_0 - t)^{1/2 + i\alpha}}\,\psi_0(\xi), \quad \text{where} \quad \xi = \frac{r}{\sqrt{t_0 - t}}. \tag{2}$$

This equation describes, in the limit $t \to t_0$, the formation of a singularity, $|\psi|^2 \to c/r^2$, at the point $r = 0$. This singularity is amenable to integration only when $d > 2$. This circumstance raises the hope that a physically reasonable solution describing a collapse can be found. Calculations carried out on a computer show that when $d = 3$, such a solution can be found if $\alpha = 0.54$.[1)] In the case of self-similar regime (2), the collapse zone contains a formally vanishing energy, and the effective dissipation factor is, by virtue of the collapse, proportional to the radius r_0. The dissipation does, in fact, occur inside this radius [in which case Eq. (1) is no longer applicable]. For this reason, Zakharov[3] called a self-similar collapse a weak collapse, from the standpoint of Eq. (1).

We will show that there can also be another collapse regime when $d = 3$. After replacing the variables $\psi = \sqrt{n}e^{i\phi}$, we can write Eq. (1) as

$$n_t + \operatorname{div} n \nabla\phi = 0,$$
$$\phi_t + \frac{1}{2}(\nabla\phi)^2 - n = \frac{1}{2}\frac{\Delta\sqrt{n}}{\sqrt{n}}. \qquad (3)$$

Let us assume that n_0 and a_0, the characteristic initial values of the intensity and size of a wave packet, are such that the semiclassical conditions are satisfied, $n_0 \gg a_0^{-2}$. We can omit the term $\Delta\sqrt{n}/\sqrt{n}$ in Eqs. (3) in this case. The system of hydrodynamic equations derived here is consistent with a self-similar solution of the type

$$n = a^{-3}(t) f\left(\frac{r}{a(t)}\right), \qquad \phi = \lambda^2 \int_0^t \frac{dt}{a^3} + \frac{a_t r^2}{2a},$$

$$f(\xi) = \begin{cases} \lambda^2\left(1 - \dfrac{\xi^2}{\xi_c^2}\right), & \xi < \xi_c \\ 0 & \xi > \xi_c . \end{cases} \qquad (4)$$

Here λ and ξ_c are arbitrary constants, and $a(t)$ satisfies the Newton equations

$$a_{tt} + \frac{dV}{da} = 0, \quad \text{and} \quad V(a) = -\frac{1}{6}\frac{\lambda^2}{\xi_c^2} a^{-3}, \qquad (5)$$

which describe the central incidence of a classical particle. If $a = 0$ at $t = t_0$, then as $t \to t_0$ we have $a \sim (t_0 - t)^{-2/5}$; here $n(0, t) \sim (t_0 - t)^{-6/5}$.

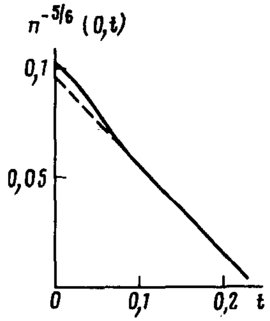

FIG. 1

Equations (4) and (5) describe a "strong" collapse of a wave packet as a whole. If we assume that all the energy concentrated in the collapse zone is absorbed there, then the efficiency of the collapse, viewed as a mechanism for wave-energy dissipation, does not depend on the size of this zone. The regime under which a collapse occurs may be called a semiclassical regime.

Since $n \sim 1/a^3$ during a semiclassical collapse, the condition under which a semiclassical regime occurs, $n \gg 1/a^2$, improves in the limit $a \to 0$. In a three-dimensional space, this circumstance is a consequence of the finite nature of the wave energy that is concentrated in the collapse zone.

A semiclassical solution of (4) and (5) does not apply to a narrow region $|\xi - \xi_c| = \Delta\xi \ll \xi_c$. As $a \to 0$, the width of this region, $\Delta\xi$, decreases in accordance with $\Delta\xi \sim a^{1/3}(\lambda\xi_c),^{-2/3}$, which is yet another indication that the semiclassical approximation can be used more effectively in the limit $t \to t_0$. A more rigorous test of this conclusion requires that the self-similar solution of (4) and (5) be joined with the solution of the linearized equation (1) for $\xi - \xi_c \gg \Delta\xi$. A boundary layer, which is described by the Penleve transcendental functions, will then appear in the zone $|\xi - \xi_c| \sim \Delta\xi$ (we will not describe this procedure because of the limitation of space).

We have solved Eq. (1) numerically in the Lagrangian variables ϵ and t ($\epsilon = \int_0^r nr^2 dr$), using a procedure similar to that of Ref. 4, with the initial conditions approximately equal to those used in the solution of (4) and (5), with $\Delta\phi = |\phi(\infty) - \phi(0)| \simeq 10$.

In the limit $t \to t_0$, the function $n(0, t)$ reaches its asymptotic behavior rapidly $n(0, t) = c(t_0 - t)^{-\beta}$, where $\beta = 6/5 \pm 0.03$ (Fig. 1). In this case, a finite energy is trapped, according to (4) and (5), in a collapse zone. For other initial conditions with $(\partial\phi/\partial r)|_{t=0} = 0$ we noticed that the function $n(0, t)$ behaves as $(t_0 - t)^{-1}$ within high accuracy. This result is consistent with the results of Ref. 5 and, at first glance, corresponds to the self-similar regime (2). Even in this case, however, we have noticed that a finite energy is trapped in a collapse zone, which is inconsistent with our conceptual understanding of a weak, self-similar collapse (2). The question of whether a self-similar solution of (2) is feasible thus remains open.

[1)]This result was obtained by L. N. Shchur.

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