Twistors and instantons in higher dimensionalities

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(Submitted 4 January 1985)
Pis'ma Zh. Eksp. Teor. Fiz. **41**, No. 4, 165–168 (25 February 1985)

Seven- and eight-dimensional "octonion" generalizations of Penrose's twistor construction are proposed. Their relationship with the corresponding duality equations and with the octonion generalization of the ADHM construction are pointed out.

Modern Kaluza-Klein theories are attracting interest to solutions of the classical equations in dimensionalities >4, in particular, analogs of the $4D$ duality equations.[1–4] For example, in a space with $4k$ dimensions, a generalization of the ADHM construction[5] has been found[4] for equations of this type[2] which lower $SO(4k)$ to $Sp(1)\times Sp(k)/Z_2$. In dimensionality 8 (and 7), however, there is still an "exceptional"—in the sense of a relationship with the algebra of octonions—Spin(7)-covariant (and thus G_2-covariant) duality. This duality is widely used in the supergravity context,[6] while the corresponding equations for a Yang-Mills field have so far been solved only for a highly simplified ansatz.[3] It is pertinent to recall that in the $4D$ case progress was made in the study of duality equations (and of related equations)[5] by using a twistor stratification

$$\mathbf{CP}^1 \to \mathbf{CP}^3 \to S^4 \tag{1}$$

of the $3D$ complex projective space $\mathbf{CP}^3$ on the 4-sphere S^4 with a $\mathbf{CP}^1$ layer. In the present letter we show how the use of octonions makes it possible to construct "truncated" [Spin(7)-covariant but not Spin(8)-covariant!] twistors suitable for an $8D$ duality and, apparently, an octonion generalization of the ADHM construction.

1. We begin with the construction, in octonion terms, of a twistor stratification[1)]

$$Q_6 \to T_{10} \to S^8 \tag{2}$$

of a complex projective manifold T_{10} on S^8 with a $6D$ complex quadric Q_6 in the layer. We will then find truncated twistors through a formal modification of this approach.

TABLE. I.

	E_0	E_b	$\bar{E}^0$	$\bar{E}^b$
E_0	E_0	E_b	0	0
E_a	0	$\epsilon_{abc}\bar{E}^c$	E_a	$-\delta_a^b E_0$
$\bar{E}^0$	0	0	$\bar{E}^0$	$\bar{E}^b$
$\bar{E}^a$	$\bar{E}^a$	$-\delta_b^a \bar{E}^0$	0	$\epsilon^{abc}E_c$

We identify $\mathbf{R}^8$ with the algebra of octonions,[7] $\mathbf{O}$, and we assume that this algebra is in turn nested in $\mathbf{C}\otimes_{\mathbf{R}}\mathbf{O}$. This is an alternative algebra[7] with divisors of zero, in which we choose a basis[8] written in the form of a column vector f,

$$f^t \equiv (E_\mu, \bar{E}^\mu) \equiv (E_A),\ \mu = 0, \dots, 3, \quad A = 0, \dots, 7, \tag{3}$$

and which has the multiplication table shown in Table I. Corresponding to a point $x\in\mathbf{R}^8$ with the coordinates $(x_M) = (x_0, x_a, x_{a+3}, x_7)$, $a = 1,2,3$, $M = 0,\dots,7$, is an element from $\mathbf{C}\otimes_{\mathbf{R}}\mathbf{O}$ of the type $x = (x^\mu, \bar{x}_\mu)f \equiv x^A E_A$, $A = 0,\dots,7, \mu = 0,\dots,3$, where

$$x^0 = x_0 + ix_7, \quad x^a = x_a + ix_{a+3}, \quad \bar{x}_\mu = \overline{x^\mu}, \tag{4}$$

and the superior bar denotes the *complex* conjugate (here $i\in\mathbf{C}\not\subset\mathbf{O}$!).

We assume $\xi,\eta\in\mathbf{C}\otimes\mathbf{O}$, with $\eta = x\xi$. The idea behind the twistor approach is to construct $x\in\mathbf{O}$ from given ξ and η. This can be done, however, if and only if ξ and η satisfy several conditions. Extending the scalar product[7] $\langle,\rangle$ from $\mathbf{O}$ to $\mathbf{C}\otimes\mathbf{O}$ in accordance with $\mathbf{C}$-linearity, we require (first) $\langle\xi, \xi\rangle = 0$. Consequently, ξ determines (in terms of uniform coordinates) a point on the $6D$ complex quadric $\overline{Q}{}_6^{\#} \subset \mathbf{CP}^7 = \mathbf{P}(\mathbf{C}\otimes\mathbf{O})$. Assuming that for a given $x\in\mathbf{O}$ we have $\eta = x\xi$, we can then write (the asterisk means *octonion* conjugation,[7] which does not affect $i\in\mathbf{C}$)

$$\eta\xi^* = 0, \quad \langle\eta,\eta\rangle = 0, \quad \langle\xi,\xi\rangle = 0. \tag{5}$$

Consequently, η determines a point of some other quadric $Q_6^{\#}$.

The triplicity principle[7] for the SO(8) group asserts that for $\omega\in so(8)$ there exist unique $\overline{\omega}^{\#}$ and $\omega^{\#}\in so(8)$ which satisfy the relation

$$\omega^{\#}(xy) = (\omega x)y + x(\overline{\omega}^{\#} y), \quad x, y\in\mathbf{O}, \tag{6}$$

and, furthermore, $\omega\to\overline{\omega}^{\#}$ and $\omega\to\omega^{\#}$ are (spinor) representations of the Lie algebra $so(8)$. Consequently, the $so(8)$-invariance of conditions (5) is ensured, since ξ and η transform by the representations specified in the notation of the corresponding quadrics. (Strictly speaking, a continuation of the representations on the basis of $\mathbf{C}$-linearity from $\mathbf{O}$ to $\mathbf{C}\otimes\mathbf{O}$ is implied.)

Among relations (5), there are five independent relations which determine the complex *manifold of twistors* $T_{10}\subset\mathbf{CP}^{15}$. If (in terms of uniform coordinates) we have $(\eta,\xi)\in T_{10}$ and $\xi\neq 0$, then an $x\in\mathbf{O}$ such that $\eta = x\xi$ is determined unambiguously. Here $\overline{Q}{}_6^{\#}$ is in a layer above x. An infinitely remote point and the layer above it are $(Q_6^{\#}, \xi = 0)$. We have thus established (2).

The relation $\eta\,\xi^* = 0$ may be interpreted in terms of a *triplicity on complex quadrics*,[9] according to which there is a set of correspondences of the type (point) $\leftrightarrow$ (3-space) $\leftrightarrow$ (3-space) between three $6D$ quadrics. (On a quadric of dimensionality 6, there are two, α and β, systems of linear subspaces of dimensionality 3. For $\xi\in\bar{Q}_6^{\#}$, those $\eta\in Q_6^{\#}$ for which $\eta\xi^* = 0$ define such a 3-space in terms of uniform coordinates.)

2. The *manifold of truncated twistors* T_9 is also determined by relations (5), but with $\xi\in\mathbf{C}\otimes\mathbf{O}'$, where $\mathbf{O}' = \{x\in\mathbf{O}|x^* = -x\}$ is a subspace of purely imaginary octonions. Choosing in $\mathbf{C}\otimes\mathbf{O}'$ the basis $h^t = (E_a,\bar{E}^a,\bar{E}^0 - E_0)$, assuming $\xi = (z^a,y_a,w)h$ and $\eta = x\xi = (v^\mu,u_\mu)f$, and using Table I, we can calculate the matrix X which represents the operator L_x, a multiplication from the left by $x[L_x:\mathbf{C}\otimes\mathbf{O}'\to\mathbf{C}\otimes\mathbf{O}]$:

$$(v,u)f = (z,y,w)Xf, \tag{7}$$

$$X = \begin{pmatrix} 0 & x^0\,\mathbf{1} & -\bar{\mathbf{x}} & -[x] \\ -\mathbf{x} & -[\bar{x}] & 0 & \bar{x}_0\,\mathbf{1} \\ -x^0 & \mathbf{x}^t & \bar{x}_0 & -\mathbf{x}^+ \end{pmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}, \tag{8}$$

where $[x]$ is the 3×3 matrix $[x]_{ab} = \epsilon_{acb}x^c$, $\mathbf{1}$ is the unit 3×3 matrix, and $\mathbf{x}^+ = \bar{\mathbf{x}}^t$. In more detail, relations (7) are

$$\begin{aligned} &v^0 = -wx^0 - y_a x^a, \qquad v^a = wx^a + z^a x^0 - \epsilon^{abc}y_b\bar{x}_c, \\ &u_0 = w\bar{x}_0 - z^a\bar{x}_a, \qquad u_a = -w\bar{x}_a + y_a\bar{x}_0 - \epsilon_{abc}z^b x^c, \end{aligned} \tag{9}$$

and relations (5) are

$$\begin{aligned} &\epsilon_{abc}v^b z^c + u_0 y_a - wu_a = 0, \qquad \epsilon^{abc}u_b y_c + v^0 z^a + v^a w = 0, \\ &v^b y_b + v^0 w = z^b u_b - wu_0 = v^\mu u_\mu = z^b y_b - w^2 = 0. \end{aligned} \tag{10}$$

Here SO(7) acts on $\mathbf{O}'$, while Spin(7) acts on $\mathbf{O}$. According to the corresponding triplicity principle,[7] for $\omega\in so(7)$ an $\omega^b\in\mathrm{spin}(7)\subset so(8)$ is determined unambiguously such that for any $x\in\mathbf{O}, a\in\mathbf{O}'$ we have

$$\omega^b(xa) = (\omega^b x)a + x(\omega a),$$

and $\omega\to\omega^b$ is a representation. This time, therefore, ξ, x, and η carry, respectively, the representations ω, ω^b, and ω^b of the Lie algebra $so(7)$. By analogy, T_9 is interpreted in terms of a triplicity among the quadric Q_5 and the two quadrics Q_6^b.

3. The $so(7)$-covariant duality equations which are of interest to us are

$$\frac{1}{2}f_{MNKL}F_{(M)(N)} = \lambda F_{(K)(L)}, \qquad M,\ldots = 0,\ldots,7, \tag{11}$$

where $F_{(M)(N)} = [\nabla_{(M)},\nabla_{(N)}]$, $\nabla_{(N)}$ represents covariant differentiation, and the components of the completely antisymmetric tensor f_{MNKL} are expressed in terms of the structure constants of the algebra[10] $\mathbf{O}$. In (11), the values $\lambda_1 = -3$ and $\lambda_2 = 1$ are allowed. Assuming

$$(\nabla_A) = (\nabla_\mu,\bar{\nabla}^\mu), \quad \nabla_0 = \nabla_{(0)} - i\nabla_{(7)}, \quad \nabla_a = \nabla_{(a)} - i\nabla_{(a+3)}, \tag{12}$$

and $F_{AB} = [\nabla_A, \nabla_B]$, we find that with $\lambda = 1$ Eqs. (11) are equivalent to the system of equations

$$2F_0{}^a = \epsilon^{abc} F_{bc}, \quad F_0^0 = F_n{}^n, \quad 2F^0{}_a = \epsilon_{abc} F^{bc}, \tag{13}$$

while with $\lambda = -3$ they are equivalent to the system

$$F_{0a} = F^{0a} = 0, \quad 4F_a{}^b = \delta_a^b (F_n{}^n - F_0{}^0),$$
$$2F_0{}^a = -\epsilon^{abc} F_{bc}, \quad 2F^0{}_a = -\epsilon_{abc} F^{bc}. \tag{14}$$

4. In $\mathbf{R}^8 \approx \mathbf{O}$, where vectors and covectors are identified, the gauge field $A_{(M)}$ corresponds to a 7×8 matrix that acts on twistors [see (7) and (8)]. In terms of a covariant differentiation, we introduce, along with (12), the matrix $\widetilde{\nabla} = (\nabla_\alpha{}^A), \alpha = 1,...,7$, with components found from (8) through the substitution $x^A \to \overline{\nabla}^A \equiv \overline{\nabla_A}$. We assume $\nabla_\alpha{}^A = \nabla_B \mathscr{E}_\alpha^{BA}$; we then have $\mathscr{E}_\alpha^{AB} = -\mathscr{E}_\alpha^{BA}$. Raising and lowering the indices $A,B,...$ and $\alpha,\beta,...$ by means of the corresponding matrices P_{AB} and $\mathscr{K}_{\alpha\beta}$, where

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathscr{K} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{15}$$

(so that, for example, $\mathscr{E}_B^{\alpha A} = \mathscr{K}^{\alpha\beta} P_{BC} \mathscr{E}_\beta^{CA}$), we find

$$N_{\alpha\beta}^{AB} = \mathscr{E}^A_{\alpha C}\, \mathscr{E}^{BC}_\beta - \mathscr{E}^B_{\alpha C}\, \mathscr{E}^{AC}_\beta . \tag{16}$$

We then have the following identities:

$$F_{AB} = \mathscr{E}_{AB\beta} F^\beta + N_{AB\alpha\beta} F^{\alpha\beta}, \quad F^\beta = \frac{1}{4} \mathscr{E}^{\beta AB} F_{AB}, \quad F^{\alpha\beta} = \frac{1}{8} N^{\alpha\beta AB} F_{AB}, \tag{17}$$

$$\mathscr{E}_{\gamma AB} N_{\alpha\beta}^{AB} = 0. \tag{18}$$

The conditions that F^α and $F^{\alpha\beta}$ vanish are equivalent to systems (13) and (14), respectively. Expansion (17) generalizes the corresponding expansion of $F_{\mu\nu}$ in $\mathbf{R}^4$ in spinor coordinates to self-dual and anti-self-dual parts. Does the analogy extend to the ADHM construction? We do not see how it would be possible to satisfy the relations $F^{\alpha\beta} = 0$ in the ADHM spirit. (A similar effect was observed in Ref. 4.) For the equations $F^\alpha = 0$, however, orthogonality condition (18) means that we can expect to be able to construct an octonion analog of the ADHM construction by working from the expression

$$\Delta^A = a^A + b^\alpha \mathscr{E}_\alpha^{AB} \widetilde{x}_B , \tag{19}$$

which is subordinate to certain quadratic relations.

5. Most of the arguments above can be extended, with the appropriate modifications, to the case of the duality equations in $\mathbf{R}^7$, which lower SO(7) to G_2. Here we should begin with a matrix representation of a G_2-covariant action $x \in \mathbf{R}^7 \approx \mathbf{O}'$ by means of a commutator on $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{O}'$. In an expanded version of this paper, we will go into the details of this construction, and we will also discuss a geometric interpretation ("geometric" here means in terms of truncated twistors and a triplicity on quadrics) of the

equations. We will describe some nontrivial compactifications of the spaces $\mathbf{R}^8$ and $\mathbf{R}^7$ which arise from the truncated twistors.

After completing this paper, I learned of Ref. 11, which has some intersection with the present paper. It is interesting to note that the Atiyah-Hitchin twistor stratification studied in Ref. 11, $\mathbf{CP}^3 \to Q_6 \to S^6$, also has a purely octonion interpretation[8]: This is a stratification of a grassmannian of oriented 2-planes in $\mathbf{O}$ on a 6-sphere of complex structures on $\mathbf{O}$.

I wish to thank V. Ya. Faĭnberg for interest in this study and for useful discussions. I am also indebted to O. V. Ogievetskiĭ for calling my attention to Ref. 11 and to R. É. Kallosh for a familiarization with Ref. 9.

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Translated by Dave Parsons