$N = 3$ harmonic superspace

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A complete solution of $N = 3$ constraints which satisfies the Bianchi identities is found. A tensor analysis of the Yang-Mills supersymmetry theory in harmonic superspace is described.

The study of expanded-supersymmetry theories is important because they are considered to play a prominent role in the derivation of the unified field theory. During the past year, there has been considerable progress made in this field because of the ability to describe geometrically the $N = 2$ and $N = 3$ Yang-Mills supersymmetry theories.[1,2] This description is based on the concept of a harmonic superspace or a space with additional dimensions.[3] Results of particular importance were obtained by Galperin *et al.*[2] They were first to construct an explicitly supersymmetric action of the $N = 3$ Yang-Mills theory which depends on the analytic superfields, the connections for the harmonic derivatives.

Before a complete geometric description of this theory could be rendered, all $N = 3$ constraints, i.e., the equations for the geometric quantities restricting the number of independent fields in the theory, had to be identified. These constraints then had to be solved and a tensor analysis of this theory in the harmonic superspace had to be formulated. In this letter we describe the solution of this problem.

We will use the analytic basis[2] in which the elements of space are $Z^M = \{z^m = \{x^{\alpha\dot{\alpha}}_{\rm an}, \theta^{(a,b)}_{\alpha}, \bar{\theta}^{(-a,-b)}_{\dot{\alpha}}, u_i^{(-a,-b)}, \bar{u}^{(a,b)i}\}, u\bar{u} = \det u = 1$. The theory of $N = 3$ flat harmonic superspace is described with the help of the covariant derivatives $D_A = \{D_{\alpha\dot{\alpha}}, D^{(-a,-b)}_{\alpha}, \bar{D}^{(a,b)}_{\dot{\alpha}}, D_r\}$ where the eight harmonic derivatives are $D_r = \{D^{(1,3)}, D^{(-1,3)}, D^{(2,0)}, D^{(-1,-3)}, D^{(1,-3)}, D^{(-2,0)}, D^{(0,0)}_1, D^{(0,0)}_2\}$. The first three derivatives are given in Ref. 2, the second three derivatives are their complex conjugates [which take $\overline{u_i^{(-a,-b)}} = \bar{u}^{(-a,-b)i}$ into account], and the last two derivatives [those of the 2U(1) generator] are obtained by commuting $D^{(-1,3)}$ with $D^{(1,-3)}$ and $D^{(2,0)}$ with

$D^{(-2,0)}$, respectively. In the flat superspace there are nonvanishing components of the torsion tensor which is determined by the relation $[D_A, D_B\} = T^C_{AB} D_C$:

$$T_{rs}^{\ \ t} = f_{rs}^{\ \ t}, \qquad T^{(a,b)(a_1,b_1)\beta\dot{\beta}}_{\alpha\ \ \dot{\alpha}} = \delta^\beta_\alpha \delta^{\dot{\beta}}_{\dot{\alpha}} \delta^{a,-a_1} \delta^{b,-b_1}, \tag{1}$$

$$T^{(a,b)(a_1 b_1)(a_2,b_2)\beta}_{\alpha} = \delta^\beta_\alpha \delta^{a+a_1, a_2} \delta^{b+b_1, b_2}, \tag{2}$$

where f^t_{rs} are the structural constants of the SU(3) group. All the remaining components of the torsion tensor, except (1) and (2), vanish.

The Yang-Mills theory can be defined by introducing the connection A_A i.e., the covariant derivative $\mathscr{D}_A + iA_A$,

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}^{\ \ C} \mathcal{D}_C + F_{AB} \tag{3}$$

with the transformation law

$$\mathcal{D}'_A = e^{i\lambda} D_A e^{-i\lambda}, \tag{4}$$

where λ is an analytic superfield. In a λ geometry, the analyticity condition means[2] that

$$\mathcal{D}^{(1,1)}_\alpha \lambda = D^{(1,1)}_\alpha \lambda = 0 \quad \text{and} \quad \bar{\mathcal{D}}^{(0,2)}_{\dot{\alpha}} \lambda = \bar{D}^{(0,2)}_{\dot{\alpha}} \lambda = 0. \tag{5}$$

Working from the two arbitrary analytic superfields, we can now construct all the components of the λ covariant derivative $\mathscr{D}_A$ of the harmonic connections,

$$A \equiv A^{(-1,3)}, \quad \overset{*}{\bar{A}} \equiv A^{(2,0)}, \quad D^{(1.1)}_\alpha A^{(-1,3)'}_{(\ 2,0)} = \bar{D}^{(0,2)}_{\dot{\alpha}} A^{(-1,3)}_{(\ 2,0)} = 0. \tag{6}$$

We determine the bounded prepotentials v and $\overset{*}{\bar{v}} (v \neq \overset{*}{\bar{v}})$

$$\begin{aligned} A^{(-1,3)} &\equiv -ie^{iv}(D^{(-1,3)} e^{-iv}), \\ A^{(2,0)} &\equiv -ie^{i\overset{*}{\bar{v}}}(D^{(2,0)} e^{-i\overset{*}{\bar{v}}}). \end{aligned} \tag{7}$$

The prepotentials v and $\overset{*}{\bar{v}}$, in contrast with the $N=2$ theory and the $N=3$ on-mass-shell theory, are determined to within certain functions that partially depend on the harmonic variables. Systems in which this arbitrariness vanishes can, however, be constructed; i.e., the dependence on the original unbounded potentials $A^{(-1,3)}_{(-2,0)}$ is unambiguous[1]:

$$\begin{aligned} A^{(1,-3)} &= -ie^{iv}(D^{(1,-3)} e^{-iv}) \\ A^{(-2,0)} &= -ie^{i\overset{*}{\bar{v}}}(D^{(-2,0)} e^{-i\overset{*}{\bar{v}}}). \end{aligned} \tag{8}$$

The remaining four harmonic connections can be constructed by using the so-called "conventional" constraints

$$F^{(-1,3)(2,0)} = F^{(1,-3)(-2,0)} = F^{(-1,3)(1,-3)} = F^{(2,0)(-2,0)} = 0, \tag{9}$$

which determine $A^{(1,3)}$, $A^{(-1,-3)}$, $A_1^{(0,0)}$, and $A_2^{(0,0)}$ in terms of the already constructed connections

$$[\mathcal{D}^{(-1,3)},\ \mathcal{D}^{(2,0)}] \equiv \mathcal{D}^{(1,3)};\ [\mathcal{D}^{(1,-3)},\ \mathcal{D}^{(-2,0)}] \equiv \mathcal{D}^{(-1,-3)};$$
$$[\mathcal{D}^{(-1,3)}_{(2,0)},\ \mathcal{D}^{(-1,-3)}_{(-2,0)}] \equiv \mathcal{D}^{(0,0)}_{\frac{1}{2}}. \tag{10}$$

Let us now construct the spinor derivatives.[2] Two of these derivatives, $A_\alpha^{(1,1)}$ and $\bar{A}_{\dot\alpha}^{(0,2)}$, vanish in the λ geometry, resolving the "remaining parts" of the constraint[2]

$$F_{\alpha}^{(1,1)}{}_{\beta}^{(1,1)} = F_{\alpha}^{(1,1)}{}_{\dot\alpha}^{(0,2)} = F_{\dot\alpha}^{(0,2)}{}_{\dot\beta}^{(0,2)} = 0. \tag{11}$$

To determine the remaining four derivatives $A_{\alpha,\dot\alpha}^{(a,b)}$, we impose conventional constraints of the type

$$F_{\alpha}^{(1,1)(-2,0)} = F_{\alpha}^{(1,1)(-1,-3)} = F_{\dot\alpha}^{(0,2)(1,-3)} = F_{\dot\alpha}^{(0,2)(-1,-3)} = 0. \tag{12}$$

As a result, we obtain the spinor connections as functions of the quantities determined above:

$$[D_\alpha^{(1,1)},\ \mathcal{D}^{(-2,0)}_{(-1,-3)}] \equiv \mathcal{D}^{(-1,1)}_{(0,-2)\,\alpha};\ [\bar{D}_{\dot\alpha}^{(0,2)},\ \mathcal{D}^{(1,-3)}_{(-1,-3)}] \equiv \mathcal{D}^{(1,-1)}_{(-1,-1)\,\dot\alpha}. \tag{13}$$

Finally, to determine the vector connection $A_{\alpha\dot\alpha}$, we impose the conventional constraint

$$F_{\alpha}^{(1,1)}{}_{\dot\alpha}^{(-1,-1)} \equiv F_{\alpha}^{(0,-2)}{}_{\dot\alpha}^{(0,2)} = 0, \tag{14}$$

which gives

$$\{\mathcal{D}_\alpha^{(1,1)},\ \mathcal{D}_{\dot\alpha}^{(-1,-1)}\} \equiv \{\mathcal{D}_{\dot\alpha}^{(0,-2)},\ \mathcal{D}_\alpha^{(0,2)}\} \equiv \mathcal{D}_{\alpha\dot\alpha} \tag{15}$$

We thus see that the connection A_A, whose components are determined in (6–8), (10), (13), and (15), is a functional of the unbounded analytic superfields A and $\overset{*}{A}$ and of their derivatives. These connections also contain and A, $\overset{*}{A}$ coupling, which is nonlocal in u space, through the integral operators of the type $(D_{(2,0)}^{(-1,3)})^{-1} A_{(2,0)}^{(-1,3)}$, which are incorporated through v and $\overset{*}{v}$.

All nonvanishing stress components, with the exception of vanishing F_{AB} components indicated above, are functionals of $A_{(2,0)}^{(-1,3)}$, $(D_{(2,0)}^{(-1,3)})^{-1} A_{(2,0)}^{(-1,3)}$, and of their derivatives. These are the λ tensors of dimensionality zero $F_{rs} \neq 0$, i.e., $F^{(-1,3)(1,3)} = W^{(0,6)}$, $F^{(2,0)(1,3)} = W^{(3,3)}$, $F^{(1,-3)(-1,-3)} = W^{(0,-6)}$, etc., with the exception of (9), the λ tensors of dimensionality 1/2, with the exception of (12), i.e., the tensors $F_\alpha^{(1,1)(1,-3)} = W_\alpha^{(2,-2)}$, $F_{\dot\alpha}^{(0,2)(-2,0)} = W_{\dot\alpha}^{(-2,2)}$, etc., and, finally, the λ tensors of dimensionality 1, except (11) and (14); e.g., $F_\alpha^{(1,1)}{}_\beta^{(-1,1)} = \epsilon_{\alpha\beta} W^{(0,2)}$, $F_\alpha^{(1,1)}{}_\beta^{(1,-1)} = W_{\alpha\beta}^{(2,0)}$. The reason for the functional dependence between the tensors is that all the tensors are functionals of A and $\overset{*}{A}$.

Let us now consider the λ geometry of the $N=3$ Yang-Mills supersymmetry theory on the mass shell. This situation implies the addition of several dynamic constraints to the kinematic constraints given above. All these constraints, taken, collectively, are

$$F_{rs} = F_{r\ \alpha}^{\ (a,\ b)} = F_{r\ \dot{\alpha}}^{\ (a,\ b)} = F_{\ \alpha\ \ \dot{\beta}}^{(a,\ b)(a_1,\ b_1)}$$

$$- \frac{1}{3}\delta^{a,\ -\ a_1}\delta^{b,-\ b_1}\sum_{(a,b)} F_{\ \alpha\ \ \dot{\alpha}}^{(a,\ b)(-\ a,\ -\ b)}$$

$$= F_{\ \alpha\ \ \beta}^{(a,\ b)(a_1,\ b_1)} + F_{\ \alpha\ \ \beta}^{(a_1,\ b_1)(a,\ b)} = 0\ . \qquad (16)$$

We can thus write the λ tensors which are nonvanishing on the mass shell as $F_{\ \alpha\ \ \beta}^{(a,b)(a_1,b_1)}$ and $F_{\ \dot{\alpha}}^{(a,b)(a_1,b_1)}$, where $a - a_1 \neq 0$ or $b - b_1 \neq 0$. The Bianchi identities, which are satisfied by imposing (16), are

$$\mathcal{D}_r F_{\ [\alpha\ \ \beta]}^{(a,b)(a_1,\ b_1)} - T_r^{\ \gamma(c,\ d)(a,\ b)} F_{\ [\alpha\ \ \gamma\ \ \beta]}^{(c,\ d)(a_1,\ b_1)} = 0. \qquad (17)$$

These identities appear in a much simpler form in the central basis where all $T_{r\ \ \alpha}^{\beta(c,d)(a,b)} = 0$.

$$\mathcal{D}_r F_{\alpha\beta}^{[ij]} = 0\ . \qquad (18)$$

The covariant independence of the harmonic directions of the physical stress tensor $F_{\alpha\beta}^{[ij]}$ in the λ geometry thus accounts for the internal consistency of the theory.

The assertions made in this study will be completely substantiated and analyzed in a separate paper.

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1) Note that the λ-covariant $\mathscr{D}^{(a,b)}$ constructed in this manner are not equal to $\overline{\mathscr{D}^{(-a,-b)}}$.

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[2] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, and E. Sokatchev, Joint Institute for Nuclear Research, Preprint E2-84-441, Dubna, 1984.
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