A new mechanism for the formation of solitons in a nematic liquid crystal

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Allowance for higher-order gradient terms in the free energy of nematic liquid crystals gives rise to a nonlinear equation of motion for the director which allows a soliton solution.

The nonlinear dynamics of liquid crystals, which leads to the formation of soliton structures in the alignment of the director, are now being studied. The formation of such structures can easily be observed experimentally by using optical methods.

Lin Lei *et al.*[1] proposed a qualitative theory to explain the soliton structure, observed experimentally by Guozhen,[2] which is produced in the director alignment of a nematic liquid crystal under the influence of a uniform shear flow. The nonlinear dynamics of the director of a nematic, which is in a static magnetic field and which is excited by an electric or magnetic pulse, was studied by Kamenskiĭ.[3] The author showed that a certain combination of parameters gives rise in this case to a soliton-type director alignment.

In the present letter we propose a new mechanism for the formation of solitons, which is based on the use of higher-order terms in the expansion of free energy in gradients of the director **n**. The fact that higher-order terms must be used in other systems in many cases was demonstrated in Refs. 4–6. The results of our study can, after some modifications, be used to describe the nonlinear dynamics of superfluid He^3 and of magnetic systems. The derivation of the equations of motion for the director **n** of a nematic liquid crystal is usually restricted to the use of free-energy expansion terms which are quadratic in the spatial derivatives. Since the energy must be invariant with respect to the transformation of the $D_{\infty h}$ group, we must stipulate that the higher-order gradient terms be also invariant with respect to the transformation of this group. We restrict ourselves to the use of fourth-order invariants $(\partial^2_{ik}\mathbf{n})^2$, $(\partial_i\mathbf{n})^4$, $(\partial_i\mathbf{n}\partial_k\mathbf{n})^2$, etc. Furthermore, we assume that this problem is effectively one dimensional (i.e., all changes in **n** are uniform in the plane perpendicular to an x axis). Such a formulation of the problem was discussed in Ref. 3. Its advantage is that it eliminates the so-called return flow.

The equation of motion for the director $\mathbf{n} = (0, \sin\varphi, \cos\varphi)$ in this case is

$$\ddot{\varphi} + \gamma\dot{\varphi} = c^2\,[\varphi_{xx}(1 + \alpha\varphi_x^2) + \beta\varphi_{xxxx}]. \tag{1}$$

Here $c^2 = K/J$, $\gamma = \gamma_1/J$, $\alpha = A/K$, $\beta = B/K$, K is the Frank rotational constant, J is the moment of inertia of the director, γ_1 is the rotational viscosity, and A and B are the appropriate combinations of the coefficients multiplying the fourth-order invariants in the free energy.

Let us assume that at time zero an initial perturbation occurs in a certain region of the sample. In this perturbation (which is uniform in the y, z plane and which has a scale dimension L along the x axis) $\varphi(t=0)=0$ and $\dot{\varphi}(t=0)=2g_0$. If the condition $g_0 \gg \gamma$ in Eq. (1) is satisfied, we can drop the dissipative term. If the initial perturbation is such that $\alpha g_0^2/c^2$ and $\beta/L^2 \ll 1$, we can omit the nonlinear and dispersion terms in (1). As a result, the initial perturbation decays in a time $t \sim L/c$ into two plane waves $\varphi = f_1(x+ct) + f_2(x-ct)$ which can now be treated independently.

Assuming that the initial perturbation is symmetrical with respect to x, let us trace the evolution of a wave that travels to the right (a complete solution is symmetric with respect to the origin of coordinates). Switching to dimensionless variables $\xi = (x-ct)/L$, $\tau = |\beta| ct/2L^3$ and introducing the function $u(\xi,\tau) = |6\beta/\alpha|^{-1/2} \dfrac{\partial\varphi}{\partial\xi}$, we find for this function

$$\frac{\partial u}{\partial \tau} + \operatorname{sign}\alpha\, 6u^2 u_\xi + u_{\xi\xi\xi} \operatorname{sign}\beta = 0\,, \tag{2}$$

i.e., a modified Korteweig-de Vries equation. Using the derivative $\partial\varphi/\partial\xi$ as the initial data for this equation at $t = L/2c$, we can solve this equation by the method of the inverse-scattering problem.[7]

The signs of the constants α and β are germane to the solution of Eq. (2). If the signs of these constants are opposite, Eq. (2) does not have a soliton solution. If the signs are the same, the solution depends on both the sign of α and β and on the initial data of the problem. If $g_0 L/c$ is larger that ~ 1, a number which is determined by solving the spectral problem for $(\partial\varphi/\partial\xi)(\xi,\tau=0)$, and sign $\alpha, \beta > 0$, then the solution of (2) consists of solitons that extend in the positive direction ξ and a continuous spectrum that extends in the negative direction. If the signs of α and β are negative, the solitons and the continuous spectrum extend in opposite directions. The soliton solution of (2) is[7]

$$u_i = 2\lambda_i \operatorname{sech}(8\lambda_i^3\tau - 2\lambda_i\xi + \delta_0^i), \tag{3}$$

where λ_i are the moduli of the imaginary eigenvalues of the spectral problem, and δ_0^i are the phases that can be calculated if the interaction of solitons with the continuous spectrum and with each other is taken into account. We see from (3) that the narrowest soliton of width $\sim 1/2\lambda_{i\max}$ moves at a higher velocity, $v = 4\lambda_{i\max}^2$. Since the amplitude of the continuous spectrum decreases as $\tau^{-1/2}$ (Ref. 8), the solution will, after a certain time, be determined primarily by solitons. Assuming that the elapsed time τ is such that the solitons have separated a distance exceeding their scale dimensions, we find $\varphi(\xi)$, with allowance for the fact that $\varphi(\infty)=0$. Integrating expression (3) over ξ, we find for each soliton u_i

$$\varphi_i = 2\,|\alpha/6\beta|^{-1/2} \arctan\{\exp[8\lambda_i^3\tau - 2\lambda_i\xi + \delta_0^i]\}. \tag{4}$$

Consequently, in its final form, the function φ in x, t coordinates appears at large times as a series of steps of height $\pi(6\beta/\alpha)^{1/2}$, which are separated by narrow (of order

λ_i^{-1}) regions that propagate at velocities close to velocity c. The optical pattern corresponding to this solution is a series of bands of various intensities, separated by narrow, transitional regions.

To describe the solution completely, we must know the nature of the initial perturbation in order to calculate its scale time, it scale dimension, and the values of λ_i and δ_0^i. The results for specific cases will be reported in a more comprehensive study. Thus far, we have been discussing the initial perturbation which is quite smooth in x. Equation (1) also allows a solitary-wave solution corresponding to a concentrated initial perturbation,

$$\varphi = 2|6\beta/\alpha|^{1/2}\arctan\exp\left[\pm(x - wt)\left(\frac{w^2/c^2 - 1)}{\beta}\right)^{1/2}\right], \quad (\mathrm{sign}\,\alpha = \mathrm{sign}\,\beta). \quad (5)$$

This wave propagates at a velocity w, which is not necessarily close to c. In this case, however, an exact solution cannot be obtained with arbitrary initial data.

The signs of the constants α and β can at least be determined from experimental observation, even if our description is only in qualitative agreement with experiment.

[1]Lin Lei, Shu Changqing, and Sher Juelian, Phys. Rev. Lett. **49**, 1335 (1982).
[2]Zhu Guozhen, Phys. Rev. Lett. **49**, 1332 (1982).
[3]V. G. Kamenskiĭ, Zh. Eksp. Teor. Fiz. **87**, 1262 (1984) [Sov. Phys. JETP **60**, No. 4 (1984)].
[4]V. R. Chechetkin, Zh. Eksp. Teor. Fiz. **71**, 1463 (1976) [Sov. Phys. JETP **44**, 766 (1976)].
[5]H. J. de Vega, Phys. Rev. D **18**, 2945 (1978).
[6]M. Ashida and S. Takegi, Progr. Theor. Phys. **65**, 1 (1981).
[7]M. Wadati, J. Phys. Soc. Jap. **34**, 1289 (1973).
[8]M. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.

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