Super-self-duality

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We show that the property of self-duality (anti-self-duality) of all field intensities that comprise super-gravitation (spins 1, 3/2, 2) constitutes a simple property of trivialization of a certain super-field. The consequence of this in particular is the absence of all vacuum divergences of super-gravitation in the field of a super- or gravitational instanton.

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The theory of super-gravitation ($N=1$) may be formulated by means of three types of super-fields.[1,2] There is the real axial-vector super-field $E_{\alpha\beta}$ (first introduced in Ref. 3) which contains the Einstein tensor $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ (the Einstein multiplet); there is the scalar chiral complex super-field U, U^* which contains the scalar curvature R (the scalar multiplet); and, third, there is the chiral spinor complex super-field $W_{\alpha\beta\gamma}$, $W^*{}_{\dot\alpha\dot\beta\dot\gamma}$, where α, $\dot\alpha$ are the dotted and undotted spinor indices, respectively, that contain among their components the conformal Weil tensor covariant with respect to super-transformations[2] (the Weil multiplet). The equations of motion of super-gravitation are as follows:

$$E_{\alpha\dot{\beta}} = U = U^* = 0. \tag{1}$$

In order to study the problems of instanton physics, super-gravitation in the Euclidean regime must be studied. This may be accomplished in accordance with Hawking's ideas.[4] The essential point that must be made when continuing into the Euclidean region is that the complex conjugation operation fails to transform the spinors with dotted indices into those with un-dotted indices and vice versa, but instead the spinors are transformed independently. Thus, parameters of the type of Weil spinors Ψ_{ABCD} and $\bar{\Psi}_{A'B'C'D'}$, introduced by Penrose,[5] are spinor equivalents of self-dual and anti-self-dual combinations of the Weil tensor

$$\Psi_{ABCD} \Rightarrow c_{abcd} - \frac{i}{2}\epsilon_{ab}^{\ \ kl} c_{klcd}, \tag{2}$$

$$\bar{\Psi}_{A'B'C'D'} \Rightarrow c_{abcd} + \frac{i}{2}\epsilon_{ab}^{\ \ kl} c_{klcd}, \tag{3}$$

which in the Lorentz regime are related by the complex conjugation operation, and in the Euclidean regime become independent

$$\Psi_{ABCD} \Rightarrow c_{abcd} + \frac{1}{2}\epsilon_{ab}^{\ \ kl} c_{klcd}, \tag{4}$$

$$\Psi_{A'B'C'D'} \Rightarrow c_{abcd} - \frac{1}{2}\epsilon_{ab}^{\ \ kl} c_{klcd}, \tag{5}$$

thus giving rise to a possibility that

$$\Psi_{ABCD} \neq 0, \quad \tilde{\Psi}_{A'B'C'D'} = 0 \tag{6}$$

and conversely

$$\Psi_{ABCD} = 0, \quad \tilde{\Psi}_{A'B'C'D'} \neq 0. \tag{6'}$$

Equation (6) in conjunction with Eq. (5) indicates that the Weil tensor is self-dual, i.e.,

$$c_{abcd} = {}^*c_{abcd} = \frac{1}{2}\epsilon_{ab}^{\ \ kl} c_{klcd}. \tag{7}$$

Equation (6′) represents the condition of anti-self-duality

$$c_{abcd} = -{}^*c_{abcd}. \tag{7'}$$

The chiral super-field $W_{\alpha\beta\gamma}$ and $W^*_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ are the super-symmetrical generalizations of the Weil spinors shown in Eqs. (2) and (3),[1,2] and in the Euclidean regime $W_{\alpha\beta\gamma}$ and $\tilde{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ generalize Eqs. (4) and (5)

$$W_{\alpha\beta\gamma} = \Phi_{\alpha\beta\gamma} + \Theta^\delta W_{\alpha\beta\gamma\delta} + (\Theta\Theta)\Delta_{\alpha\beta\gamma}, \tag{8}$$

$$\tilde{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \tilde{\Phi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} + \tilde{\Theta}^{\dot{\delta}} \tilde{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + (\tilde{\Theta}\tilde{\Theta})\tilde{\Delta}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}. \tag{9}$$

In Eqs. (8) and (9) Θ^{δ} and $\tilde{\Theta}^{\dot{\delta}}$ are the Grassman variables of the super-fields (in the Euclidean regime unrelated by complex conjugation). The spinor $\Phi_{\alpha\beta\gamma}$ and $\Delta_{\alpha\beta\gamma}$ are related to the spin 3/2 field intensity and $W_{\alpha\beta\gamma\delta}$ contains in addition to the Weil spinor, i.e., gravitational field intensity, the spinor equivalent of intensity of an auxiliary axial-vector super-gravitational field A_{μ}.

The basic result of this work is that the condition of self-duality of all three intensities is as follows (compare with Ref. 6):

$$W_{\alpha\beta\gamma} \neq 0, \qquad \tilde{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = 0, \tag{10}$$

and the condition of anti-self-duality is

$$W_{\alpha\beta\gamma} = 0, \qquad \tilde{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \neq 0. \tag{11}$$

The reason why the conditions of (anti)-self-duality are so simple in terms of spinors that characterize the Riemannian parameters, is as follows. The anti-symmetric tensor with the Lorentz indices is transformed into its spinor equivalent by means of the matrices $\tilde{\sigma}^{ab}_{\dot{\alpha}\dot{\beta}}$ and $\sigma^{ab}_{\alpha\beta}$ that are characterized in terms of their structure by the property of anti-self-duality or, respectively, self-duality

$$\tilde{\sigma}^{ab}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\epsilon^{ab}_{\ \ cd}\,\tilde{\sigma}^{cd}_{\dot{\alpha}\dot{\beta}}, \tag{12}$$

$$\sigma^{ab}_{\alpha\beta} = \frac{1}{2}\epsilon^{ab}_{\ \ cd}\,\sigma^{cd}_{\alpha\beta}. \tag{13}$$

Thus, the spinor equivalent of any anti-symmetric tensor with Lorentz indices (such as the field intensities are) is, in terms of its structure, associated with the self-dual or anti-self-dual combination of Riemannian parameters.

Equations (10) and (11) have trivial (in the sense of super-symmetry) examples. These self-dual or anti-self-dual gravitational instantons have been the subject of intensive studies in recent times,[6] and are equal to the zero of the 3/2- and 1-spin field. It would be interesting to find the nontrivial solutions of Eqs. (10) and (11), i.e., super-instantons.

The interesting application of Eqs. (10) and (11) occurs in the analysis of divergence of super-gravitation. As we know, the equations of motion [Eq. (1)] may be used in the analysis of scale-invariant divergences. In this case, super-gravitation contains, in terms of single-loop approximation, only the trivial divergence that is associated with super-topological invariants[7]; in terms of the two-loop approximation the divergences are absent, and in terms of the three-loop approximation, the divergence is associated with a super-field[8,1]

$$W_{\alpha\beta\gamma}W^{\alpha\beta\gamma}\tilde{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}\tilde{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}. \tag{14}$$

The above super-field constitutes a super-symmetrical conformal generalization of the square of the conserved traceless Bell-Robinson tensor[8,2]

$$T^{\;2}_{mnab} = [R^{k\;\;p}_{\;a\;\;m} R_{kbpn} + {}^*R^{k\;\;p}_{\;a\;\;m}\, {}^*R_{kbpn}]^2. \tag{15}$$

It follows from Eqs. (8), (9), (12) and (13) that the gravitational portion of the super-field [Eq. (14)] is

$$(C_{abcd} - {}^*C_{abcd})^2 (C_{klmn} + {}^*C_{klmn})^2 \tag{16}$$

and the property of trivialization [Eq. (14)] in the field of self-dual and anti-self-dual gravitational or super-instanton—which would be extremely difficult to take into account in an expression of the type of Eq. (15)—follows in the form of Eq. (14), trivialially from Eqs. (10) or (11), or in the form of Eq. (16), from Eqs. (7) or (7′).

This detailed analysis shows that all the subsequent invariants that correspond to scale-invariant local n-loop divergences of super-gravitation, become trivial in the field of the self-or anti-self-dual instanton. In the proof it is essential to show that in Eqs. (8) and (9) $\Delta_{\alpha\beta\gamma} = \widetilde{\Delta}_{\dot\alpha\dot\beta\dot\gamma} = 0$ in the equations of motion and, therefore, [e.g., for Eq. (10)], super-fields constructed from $W_{\alpha\beta\gamma}$ and its derivatives do not contain in the D-term a contribution from the 3/2 spin in the equations of motion.

We should note that in contrast to super-gravitation, quantum gravitation does not yield such reductions of divergence in the instanton field.

We believe that the observations made in this work will stimulate interest in the topological approach to gravitation and, particularly, super-gravitation.

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