The hydrodynamics of black hole vaporization

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An approximate spherically-symmetric solution is obtained for the equations of relativistic hydrodynamics which describe the latter stage of the quantum vaporization of a black hole. The hydrodynamic state—terminating at temperatures of the order of the π-meson mass—provides means for finding the energy distribution of scattering particles.

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Hawking has shown[1] that a small black hole with mass m emits like a black body with a temperature $T = m^{-1}$.[1)] This process leads to the vaporization of a small-mass black hole in a finite time: $m = (-t)^{1/3}$ (counting time from the onset of vaporization). The emission temperature at $m < \mu^{-1}$ exceeds the π-meson mass and the produced particles interact strongly with each other. Thus, the shape of the emission pulse that propagates from the exploded black hole must be determined by solving the relativistic hydrodynamics problem which is similar to the case examined by Landau of multiple particle production.[2] The Landau solution for a one-dimensional dispersion represents a wave with a steep front and exponential decay. Therefore, when addressing the problem of spherically-symmetric dispersion one may expect that an attenuation of the radiation pulse following the black hole explosion also exhibits a certain asymptotic power behavior.

In addition to having astrophysical applications, this problem also represents a considerable methodological interest in connection with the Kundt problem,[3] for which the entropy is defined on the surface of a black hole or in its emission. The solution found in our work is adiabatic, which allows us to calculate for a finite number of dispersed particles with respect to the initial entropy of a black hole.

We shall assume that the equation of state of an ultrarelativistic medium $p = \epsilon/3$ holds without any restrictions. Thus, the energy-momentum tensor is $T_i^k = \epsilon(4u_i u^k \delta_i^k)/3$, where u_i is the four-dimensional velocity. The region where hydrodynamics applies—specified by the inequality $T > \mu$—considerably exceeds the gravitational radius of a black hole in the later stages of dispersion. We shall, therefore, write the laws of conservation $T^k_{i;k} = 0$ in a plane spherically-symmetric metric. This yields two equations for the energy density $\epsilon(r,t)$ and velocities $u_0 = u^0 = (1 + u_1^2)^{1/2}$; $u^1 = -u_1$

$$r^2 \frac{\partial}{\partial t} \epsilon (4u_0^2 - 1) + 4 \frac{\partial}{\partial r} \epsilon u_0 u^1 r^2 = 0,$$

$$4 \frac{\partial}{\partial t} \epsilon u_0 u^1 + \frac{\partial}{\partial r} \epsilon (4u_1^2 + 1) + 2\epsilon u_1^2 r^{-1} = 0. \quad (1)$$

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The foregoing equations yield

$$\frac{\partial}{\partial t}\epsilon^{3/4} r^2 u^0 + \frac{\partial}{\partial r}\epsilon^{3/4} r^2 u^1 = 0,$$

which represents the law of conservation of entropy $(su^i)_{;i} = 0$ ($s \sim \epsilon^{3/4}$—entropy density in the case of the ultrarelativistic equation of state).

Let us first consider a stage characterized by slow, quasi-stationary vaporization. If we assume that $u \gg 1$ and neglect the time-dependent derivatives in the hydrodynamic equations, we get $\epsilon = T^4 = r^{-4}$; $u = r(-t)^{-1/3}$. The region of hydrodynamic applicability extends, in this case, from the black hole surface $r = (-t)^{1/3}$ to a radius $r = \mu^{-1}$ where the temperature is equivelant to the interaction energy μ. In order that the quasi-stationary solution may apply, we need the black hole mass to be much greater than the total energy in the hydrodynamic region, i.e., $m = (-t)^{1/3} \gg \int^{\mu^{-1}} \epsilon u^2 r^2\, dr = u^{-1}(-t)^{-2/3}$. Thus, the quasi-stationary vaporization stage occurs when $m \gg m_0 = \mu^{-1/3}$.

We shall now search for a solution of Eq. (1) beyond the vaporization stage in the region $r \gg |r - t|$ assuming that $u_0 \approx u^1 \approx u \gg 1$. We introduce the following notations: $\tau = \log r/R_0$; $\eta = \log|r - t|/\xi_0$; $l = \log \epsilon/\epsilon_0$. with the constants R_0, ξ_0, and ϵ_0 still unknown, we get the following equation for the function $l(\tau,\eta)$:

$$3 + 2\frac{\partial l}{\partial \eta} + \frac{\partial l}{\partial \tau} + \frac{3}{4}\frac{\partial l}{\partial \eta}\frac{\partial l}{\partial \tau} = 0, \tag{2}$$

which differs from a similar equation of one dimensional hydrodynamics only by virtue of the constants.[2] The velocity u is found from the following expression:

$$u^2 = \frac{r}{(r-t)}\frac{1}{4}\frac{\partial l/\partial \eta}{(3 + \partial l/\partial \tau)}.$$

The general solution of Eq. (2) is in the parametric form

$$l(\tau, \eta) = A\eta - \frac{3 + 2A}{4 + 3A}\tau + B(A);$$

$$\frac{\partial l}{\partial A} = \eta + \frac{\tau}{(4 + 3A)^2} + \frac{dB}{dA} = 0;$$

where the function $B(A)$ is arbitrary. The precise formulation of the boundary condition for Eq. (2)—as also in the case of one-dimensional hydrodynamics—is difficult. We shall determine the asymptotic solution of Eq. (2) for $\eta \gg |dB/dA|$ when the shape of the propagating wave is independent of the conditions of energy released. Assuming $B = 0$—which corresponds to predetermination of ϵ_0—we get

$$l(\tau, \eta) = -\frac{4}{3}\eta - \frac{8}{3}\tau \pm \frac{4}{3}\sqrt{-\tau\eta};$$

$$u^2 = \pm\frac{r}{r-t}\sqrt{-\frac{\tau}{4\eta}}. \tag{3}$$

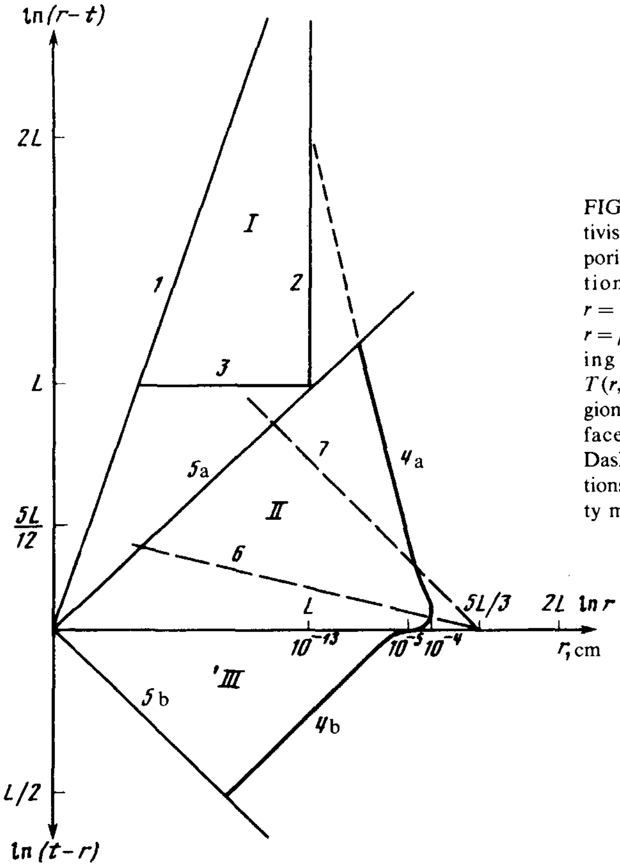

FIG. 1. Regions of applicability of the relativistic hydrodynamics solutions for the vaporization of a black hole: I—quasi-stationary vaporization; curves: 1—$r=(-t)^{-1/3}$—black hole surface; 2—$r=\mu^{-1}$; 3—$(-t)=\mu^{-1}$. II—region of leading wavefront; curve 4a—surface $T(r,t)=\mu$, curve 5a—$r-t=r$. III—region of trailing wavefront; curve 4b—surface $T(r,t)=\mu$, curve 5b—$t-r=r$. Dashed lines 6 and 7 show respective positions of the energy and entropy flux density maxima.

The positive nature of u^2 indicates that for $r>T$ (wavefront) the upper sign in Eqs. (3) applies, and for $r<t$, the lower. Thus, in the case of a spherically-symmetric hydrodynamic dispersion, as opposed to the one dimensional case, the wavefront steepens according to the power law.

In order to evaluate the constants R_0, ξ_0, and ϵ_0 we shall consider the time dependence of the energy flux $T_0^1=\epsilon u^2$ and entropy flux $su^1=\epsilon^{3/4}u$ through the surface of a sphere $4\pi r^2$. At $r>t$, if we neglect the coefficients and the logarithmically slowly varying expressions, we have

$$\epsilon u^2 r^2 d(r-t) = \exp\left\{-\frac{1}{3}\left(\sqrt{\eta}-2\sqrt{-\tau}\right)^2\right\}\epsilon_0 R_0^3\, d\eta \ ;$$

$$dN = su r^2 d(r-t) = \exp\left\{-\frac{1}{2}\left(\sqrt{\eta}-\sqrt{-\tau}\right)^2\right\}\epsilon_0^{3/4} R_0^{5/2}\xi_0^{1/2}\, d\eta\,. \quad (4)$$

In order that the integrals may converge, r must be negative, i.e., $\xi_0 \ll |r-t| \ll r \ll R_0$. The integration of expressions in Eqs. (4) leads to the result that the total energy and total number of particles (entropy)—crossing a sphere with of a given radius r—are constant quantities. They represent, respectively, the initial mass of a black hole m_0 and its initial entropy which, in accordance with Ref. 1, is m_0^2. We must pick the black hole mass for m_0 for which the quasi-stationary vaporization regime fails to hold, i.e., $m_0 = \mu^{-1/3}$. The smallest parameter of the problem ξ_0 which determines the transition from the leading wavefront to trailing, should be of the order of Planck's length: $\xi_0 = 1$. Having calculated the remaining constants, we shall formulate the final result for the coordinate-dependent temperature:

$$T = \begin{cases} \exp\left\{ -\dfrac{L}{18} - \dfrac{\ln(r-t)}{3} - \dfrac{2\ln r}{3} + \dfrac{1}{3}\sqrt{\ln(r-t)\left(\dfrac{5}{3}L - \ln r\right)} \right\}, & r > t \\ \exp\left\{ -\dfrac{L}{18} - \dfrac{\ln(t-r)}{3} - \dfrac{2\ln r}{3} - \dfrac{1}{3}\sqrt{\ln(t-r)\left(\dfrac{5}{3}L - \ln r\right)} \right\}, & r < t \end{cases}.$$

Above, $L = -\log\mu = 46$. The region of applicability of the above equation—which follows from the inequalities $T > \mu$ and $|r-t| < r$—is shown in Fig. 1. The energy density maximum (temperature maximum) and particle density maximum—both calculated from Eqs. (4)—are attained inside the region of applicability of the solution.

After crossing the surface $T = \mu$, the particles cease interacting with each other and disperse in a manner such that the energy of each particle $E = Tu = \mu[r/(r-t)]^{1/2}$ remains subsequently constant. The second relationship in Eqs. (4) may be used to find the particle energy distribution dN/dE in parametric form. This distribution exhibits a maximum at $E = \exp|-0.083\, L|$.

In conclusion, we shall address the astrophysical aspects of our solution. The shape of a pulse from an exploded black hole that a distant observer would see is determined essentially not by the hydrodynamic stage but the subsequent scattering of dispersed particles in interstellar space. Therefore, of greatest interest to astrophysics (in particular, to the determination of the formation of elements in a universe containing black holes[4]) is the study of interaction between the particle flux with a given energy distribution and the surrounding medium.

[1)] Planck's system of units is used in this article: $c = \hbar = G = 1$. All equations are written with accuracy up to constant multipliers of the order of unity. The π-meson mass μ is a parameter of the problem and in Planck's units is 10^{-20}.

[1] S.W. Hawking, Nature **248**, 30 (1974).
[2] L.D. Landau, Collected works **2**, 153, 260, M., Nauka (1959).
[3] W. Kundt, Nature **259**, 30 (1976).
[4] Ya.B. Zel'dovich, A.A. Starobinskiĭ, M.Yu. Khlopov, and V.M. Chechetkin, Pis'ma Astronom. Zh. **3**, 208 (1977) [Sov. Astron. Lett. **3**, 110 (1977)].