The theory of nuclear displacement of levels of a proton-antiproton atom

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A formula is obtained which expresses the displacement of the Coulomb levels of hadronic atoms in terms of the scattering length due to nuclear potential (without assumptions concerning the smallness of displacement). Experimental data on the shift of the $1s$-level of $p\bar{p}$-atom indicate a possibility of the existence in the $p\bar{p}$-system of a bound state with binding energy $\epsilon \lesssim 0.9$ MeV and width $\Gamma \lesssim 200$ KeV.

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As we know the nuclear interaction causes displacement of the Coulomb levels of an atom and their broadening. If the level shift ΔE_{nl} is small, perturbation theory is applicable[1,2]

$$\Delta E_{nl} = \frac{2(n+l)!}{(l!)^2 (n-l-1)! n^{2l+4}} a_l^3, \qquad (1)$$

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where a_l is the scattering length due to the nuclear potential V_{st} (the Coulomb system of units is used $\hbar = e = m = 1$, m is reduced mass). When the potential V_{st} contains a near-zero real or imaginary level, the scattering length becomes large and Eq. (1) cannot be used. As soon as a level occurs in V_{st} the atomic spectrum undergoes reconstruction[3,4] (similar behavior of the s- and p-levels was observed elsewhere[5] near the critical nuclear charge which corresponds to the entry of the $1s$ level into the lower continuum).

We shall derive a formula for the displacement of Coulomb levels which holds outside the framework of the perturbation theory and describes the effect of spectrum reconstruction. For this we use the condition $r_0 \ll a_B$, where $a_B = \hbar^2/me^2 = 1$ is Bohr's radius, $r_0 = \mu^{-1}\ln[(\mu/2m_N)^2 g^2/e^2] \approx 3.4\Phi$ is a point at which the potential V_{st} is comparable to the Coulomb potential (μ and m_N are masses of π-meson and nucleon, respectively). In the region $r \gg r_0$ a solution of the Schrödinger equation—which decreases at infinity—is expressed in terms of the Whittaker function $W_{1/\lambda, l+\frac{1}{2}}(2\lambda r)$. Expanding it when $r \to 0$ and connecting at a point $r = r_0$ with the logarithmic derivative of the nuclear wave function

$$\xi = r \frac{d}{dr} \ln (rR)\Big|_{r=r_0},$$

in the case of s-levels we get[1)]:

$$1/a_s = 2[\psi(1 - \lambda^{-1}) + \lambda/2 + (1 + 4r_0)\ln\lambda + (1 - 4r_0)\ln r_0 + c_0]. \tag{2}$$

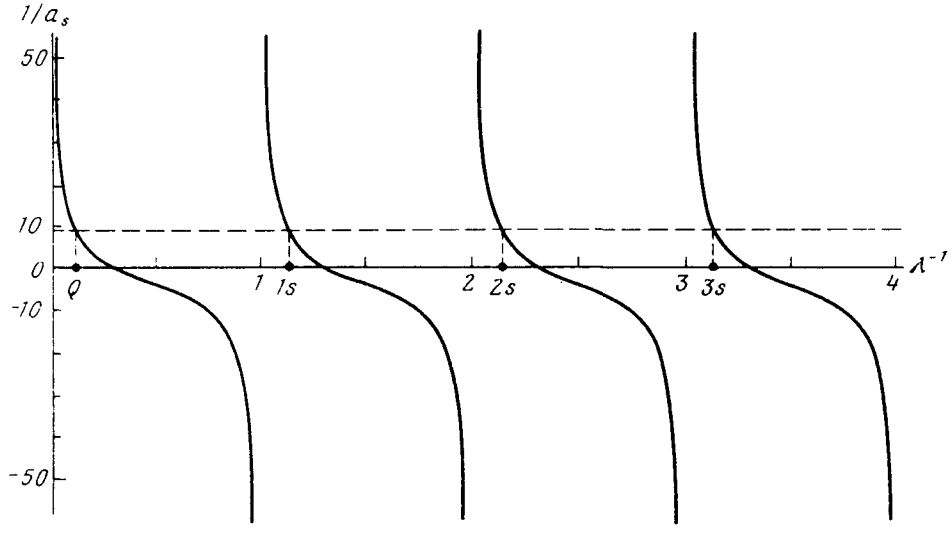

FIG. 1. Dependence of $1/a_s$ on λ^{-1} in accordance with Eq. (2) for $r_0/a_B = 0.05$. Also shown is the position of the Coulomb levels ns and the quasi-nuclear level Q (see Ref. 4) which corresponds to displacement of the $1s$-level measured in Ref. 6.

Here $E = -\lambda^2/2$ is the level energy, $\psi(z) = \Gamma'(z)/\Gamma(z)$, $c_0 = 2\gamma + \ln 2 = 1.848$ (γ is he Euler constant). In deriving Eq. (2) we used a relationship between the scattering length a_l with the parameters ξ and r_0:

$$a_l = a_l^{(0)} \frac{\xi - (l+1)}{\xi + l}, \quad a_l^{(0)} = \frac{2^{2l}(l!)^2}{(2l)!\,(2l+1)!} r_0^{2l+1} \tag{3}$$

and terms of the order of r_0/a_B are neglected.

The function represented by Eq. (2) is plotted in Fig. 1. The atomic level spectrum is defined by the points of intersection of this plot with the line $a_l = \text{const}$. We should emphasize that Eq. (2) also holds for a complex a_s and it describes level displacements in the region of applicability of perturbation theory in terms of the scattering amplitude [see Eq. (1)], and also in the region of the reconstructed atomic spectrum. In order to obtain Eq. (1) from Eq. (2) we must allow for the fact that when $|a_s| \ll 1$ we are in the vicinity of a pole of the ψ-function: $\lambda = 1/(n+\nu)$, $\nu \to 0$. Taking into account the expansion $\psi(1-n-\nu) = \nu^{-1} + \psi(n) + 0(\nu)$, we get

$$a_s \approx \frac{\nu}{2(1 - A_n \nu)}, \quad \Delta E_{n0} = \frac{1}{2n^2}\left[1 - \left(1 + \frac{2a_s}{n(1+2A_n a_s)}\right)^{-2}\right] \approx \frac{2}{n^3} a_s,$$

which is in agreement with Eq. (1). However, this approximation holds only for $|\nu| \ll 1/A_n$, where $A_n = \ln(n/r_0) - \psi(n) - 1/2n - c_0$. In view of the large value of the

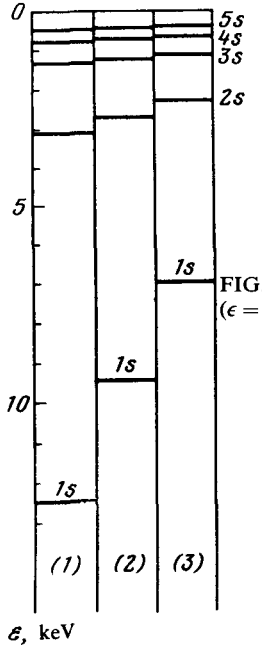

FIG. 2. Atomic spectrum of $p\bar{p}$-atom for three values of s-scattering length ($\epsilon = -E$—binding energy): 1—$a_s = 0$; 2—$a_s = 6.6\Phi$; 3—$a_s = \infty$.

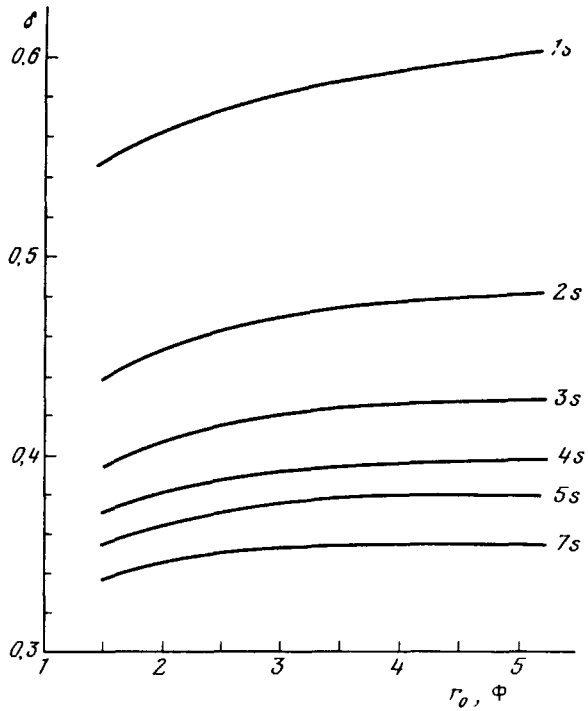

FIG. 3. Dependence of level displacement ΔE_{n^0} at $a_s = \infty$ on parameter r_0: $\delta = \Delta E_{n^0}/(E^C_{n+1} - E^C_n)$ for Coulomb levels ns.

logarithm ($\ln 1/r_0 \gg 1$) the region of applicability of the approximation [Eq. (1)] is relatively narrow.

In accordance with Eq. (2) at $a_s = 0$ we have: $\lambda = 1/n$, $E_n = E^C_n \equiv -1/2n^2$ and the spectrum is strictly Coulombic. The critical case $a_s = \infty$ corresponds to complete restructuring of the atomic spectrum and differs the most from the Coulomb spectrum (see Fig. 2). Although the connecting point r_0 figures in Eq. (2), Coulomb energy levels are logarithmic functions of r_0 and, therefore, uncertainty in the value of r_0 is immaterial (see Fig. 3). Greater dependency on r_0 is observed for a "quasi-nuclear" level Q whose binding energy far exceeds the Coulomb unit energy $me^4/\hbar^2 = 25.0$ KeV.

Equation (2) may be used to calculate the position and width of the remaining s-levels from the known displacement and width of one of the s-levels, and also to calculate the scattering length a_s. Using the results of measurement of displacement of the ground level of the $p\bar{p}$-atom[6] ($\Delta E_{1s} = 3.02 \pm 0.06$ KeV, $\Gamma \lesssim 200$ eV) we get:

$$a_{p\bar{p}} = (6.66 - i\,0.37)\,\Phi \tag{4}$$

with an error of the order of 0.3Φ in the real part of the scattering length.[2)] This means that in a given spin state ($S = 0$ or 1) the elastic cross section in the $p\bar{p}$ system at zero energy is $\sigma_{el} = 4\pi|a_{p\bar{p}}|^2 = 5.6$ bn, and the annihilation cross section[3)] is $\lim_{v\to 0} v\sigma_a =$ 19.5 m bn. It should be pointed out that the perturbation theory [Eq. (1)] yields: $a_{p\bar{p}} = (3.48 - i0.12)\Phi$ which differs substantially from Eq. (4).

Figure 1 indicates that in addition to atomic levels the $p\bar{p}$ system contains the

nuclear s-level Q. Using Eq. (2) we find its binding energy $\epsilon = 870$ KeV and width $\Gamma = 145$ KeV. Estimates using a formula for the zero effective force radius and including the value of $a_{p\bar{p}}$ from Eq. (4), yields similar results: $\epsilon = 930$ KeV and $\Gamma = 210$ KeV. Thus, if the experimental data in Ref. 6 are correct, the $p\bar{p}$ system should contain a quasi-nuclear meson[4] with mass 1875.5 MeV, width $\Gamma \lesssim 200$ KeV and spin 0 or 1.

In the case of states with the orbital moment $l \neq 0$ we get:

$$\phi_l(\lambda)\left[\psi(l+1-\lambda^{-1}) - \ln \lambda r_0\right] = a_l^{-1} + \tilde{a}_l^{-1} + \ldots, \tag{5}$$

where

$$\tilde{a}_l = \frac{2^{2l-1}(l!)^2}{(2l)!(2l-1)!} r_0^{2l}, \quad \phi_l(\lambda) = \frac{2\lambda^{2l+1}\Gamma(l+1+\lambda^{-1})}{(l!)^2\Gamma(\lambda^{-1}-l)}.$$

The plot of the left-hand side of Eq. (5) in the region $l < \lambda^{-1} < \infty$ is similar to Fig. 1. In the right-hand side the term $1/\tilde{a}_l$ represents, generally speaking, a correction $\sim r_0$ to the term $1/a_s$. However, the correction is significant for $\xi = -1$, i.e., in the region of reconstructed spectrum. The latter is determined by the condition $|\xi + l| \lesssim (r^0/a_B)^{2l+1}$ and its width decreases with increasing l. Outside this region, $\psi(l+1-\lambda^{-1})$ is near the pole and Eq. (5) changes to Eq. (1).

The relativistic Coulomb problem, for which the parameter $\zeta = Ze^2 \approx Z/137$ is not small, may be similarly considered. The Klein-Gordon or Dirac equations for $r > r_N$ have exact solutions. Coupling with an internal solution at the boundary of a nucleus yields a solution for the level spectrum which when studied (assuming $r_N \ll \lambda_e = \hbar/m_e c$) leads to the following conclusions.

1. Formulas of the type of Eqs. (2) and (5) fail to agree with the perturbations theory even at $|\nu| \ll 1$. This is associated with the fact that the cut-off of the Coulomb potential $V(r) = -\zeta/r$ inside the nucleus ($0 < r < r_N$) qualitatively alters the behavior of the wave functions at small distances: $G \sim r^\gamma$ changes into $G \sim r^{j+1/2}$, where $\gamma = [(j+\frac{1}{2})^2 - \zeta^2]^{1/2}$. At $Z = 92$, the deviation between the relativistic analogs of Eqs. (1) and (2) is $\sim 50\%$ for the level $1s$. This must be borne in mind in calculations of the π- and μ-atoms with large Z for which precision measurements of the level spectrum exist.

2. The width of the reconstructed region of the Coulomb spectrum is $\Delta\xi \sim (r_N/\lambda\!\!\!^{-}_e)^{2\gamma}$ and it broadens with increasing Z. At $\zeta_{cr} - \zeta \sim [\ln(\lambda\!\!\!^{-}_e/r_N)]^{-2}$ the power $\Delta\xi$ vanishes and the regime becomes logarithmic. In accordance with Ref. 5, spectral reconstruction of the heavy atom—when a level drops to the lower continuum—occurs in the interval $|\zeta - \zeta_{cr}| \sim [\ln(\lambda\!\!\!^{-}_e/r_N)]^{-2}$. Thus, an increase in the nuclear charge makes the spectral reconstruction effect less sudden.

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[1)]By virtue of the specifics of the expansion of the Whittaker function at $r \to 0$, the inserts at $l = 0$ and $l \geqslant 1$ are slightly different. The formula for the case $l \geqslant 1$ is given below—see Eq. (5).

[2] Positive displacement of ΔE_{1s} was first obtained in Ref. 7 by means of the numerical solution of Schrödinger's equation with OBEP potential.

[3] Since only an upper-bound for the width Γ was obtained in Ref. 6, the values shown for $\mathrm{Im}a_{p\bar{p}}$ and $v\sigma_a$ also represents the upper limits of these parameters.

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