Classical and quantum dynamics of particles with arbitrary spins

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Grassman variables are used to construct a Lagrangian that is invariant with respect to the internal local group $O(n)$ and describes the mechanics of a mass with spin $n/2$.

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The introduction to the physics of elementary particles of anti-commuting variables and the study of transformation groups with similar parameters has led to discovery of the properties of a symmetry between bosons and fermions. The intensive development of this approach in the physics of elementary particles—so called supersymmetry (see review[1])—has given a new insight into the earlier attempts to

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formulate the Hamiltonian spin mechanics for the point particle. Thus, a generalization of the classical mechanics of a particle with spin was proposed,[2,3] which is based on the expansion of the normal phase space associated with the allocation of position and speed of a particle, and the addition of the Grassman variables applicable to the spin degrees of freedom. The subsequent development of this approach was advanced on the basis of determining the supersymmetrical transformations in the aforementioned space.[4,5]

In this work we formulate the classical and quantum mechanics of a point particle with spin $n/2$ on the basis of the field approach used in supergravitation[6] with the internal local symmetry $O(n)$. In contrast to supergravitation in four-dimensional space-time, limitations on the dimensionality n of the $O(n)$ group in the description of point mechanics do not exist.

Motion of a particle with spin is described by means of the local supersymmetrical and local $O(n)$ transformations that are invariant with respect to reparametrization. The quantum theory is formulated by superposing canonical commutation relations. Moreover, relations originating in the theory and imposed on the physical state vectors lead to the Dirac equation and the mass shell condition for the Bargman-Wigner wave function of a particle with spin $n/2$.

The original Lagrangian describes the interaction of an "expanded" supergravitation (fields e_N^a, λ_N^k, and V_N^{kl}) with "supermatter" (fields X_μ and ψ_μ^k).[1)] In a space with a single time and without spatial coordinates the purely "supergravitational" part vanishes and the Lagrangian has the following form:

$$L = \frac{\dot{X}_\mu^2}{2e} - \frac{i}{2}\psi_\mu^k\dot{\psi}_\mu^k - \frac{i}{2e}\lambda^k\psi_\mu^k\dot{X}_\mu - \frac{1}{8e}(\lambda^k\psi_\mu^k)^2 + i\psi_\mu^k\psi_\mu^l V^{kl}. \tag{1}$$

The action corresponding to the above Lagrangian is invariant with respect to the reparametrization:

$$\delta e = (ae)^{\cdot}, \qquad \delta\lambda^k = (a\lambda^k)^{\cdot}, \qquad \delta V^{kl} = (a V^{kl})^{\cdot},$$
$$\delta X_\mu = a\dot{X}_\mu, \qquad \delta\psi_\mu^k = a\dot{\psi}_\mu^k, \tag{2}$$

of the local supersymmetrical transformations:

$$\delta e = i a^k\lambda^k, \quad \delta X_\mu = i a^k\psi_\mu^k, \quad \delta\lambda^k = 2\dot{a}^k + 4a^l V^{kl},$$
$$\delta\psi_\mu^k = \frac{a^k}{e}\left(\dot{X}_\mu - \frac{i}{2}\lambda^l\psi_\mu^l\right), \qquad \delta V^{kl} = 0 \tag{3}$$

and the local $O(n)$ transformations:

$$\delta e = 0, \quad \delta X_\mu = 0, \quad \delta\lambda^k = t^{kl}\lambda^l, \quad \delta\psi_\mu^k = t^{kl}\psi_\mu^l,$$

$$\delta V^{kl} = \frac{\dot{t}^{kl}}{2} + t^{kn} V^{nl} - t^{ln} V^{nk} . \tag{4}$$

We should note that the algebra of transformations (2)–(4) is closed only if the equations of motion are taken into account. Moreover, fields e, λ^k and V^{kl} which form the "supermultiplet" have no equations of motion due to the small space-time scale and, therefore, the algebra closes for these fields. The use of equations of motion for the "mass" fields X_μ and ψ_μ^k reduces to normal situation in the supersymmetry theories and corresponds to exclusion of the auxiliary fields from the "mass" supermultiplet. Inclusion of these, on the other hand, leads to closing the transformation algebra.

For example, the commutator of two supersymmetrical transformations is equal to the combination of coordinate transformations, both supersymmetric and $O(n)$ transformations:

$$[\delta_\alpha, \delta_\beta] A = \delta_a A + \delta_{\tilde{\alpha}} A + \delta_t A , \tag{5}$$

where A is any field $(e, \lambda^k, V^{kl}, X_\mu, \psi_\mu^k)$ and the transformation parameters are:

$$a = \frac{2i\alpha^k\beta^k}{e} \qquad \tilde{\alpha}^k = -i\frac{\alpha^n\beta^n}{e}\lambda^k , \qquad t^{kl} = -\frac{4i\alpha^n\beta^n}{e} V^{kl} . \tag{6}$$

The equations of motion which follow from the Lagrangian Eq. (1) are:

$$\dot{P}_\mu = 0, \qquad \dot{\psi}_\mu^k = \frac{\lambda^k}{2} P_\mu + 2\psi_\mu^l V^{kl} , \qquad P_\mu = \frac{1}{e}\left(\dot{X}_\mu - \frac{i}{2}\lambda^k\psi_\mu^k\right), \tag{7}$$

and the relations

$$P_\mu^2 \approx 0, \qquad \psi_\mu^k P_\mu \approx 0, \qquad \psi_\mu^k \psi_\mu^l \approx 0, \qquad \Pi_\mu^k - \frac{i}{2}\psi_\mu^k \approx 0 \tag{8}$$

describe the free motion of a classical massless material point with spin $n/2$. This is most graphically evident in units where $e = 1$, $\lambda^k = V^{kl} = 0$.

Quantizing this theory in the standard way (see Ref. 7), let us impose the following canonical commutation conditions:

$$[X_\mu, P_\nu]_- = i\hbar\{X_\mu, P_\nu\} = -i\hbar g_{\mu\nu},$$

$$[\psi_\mu^k, \psi_\nu^l]_+ = i\hbar\{\psi_\mu^k, \psi_\nu^l\}^* = -\hbar g_{\mu\nu}\delta^{kl}, \tag{9}$$

where $\{\ \}$ is the Poisson bracket and $\{\ \}^*$, the Dirac bracket.

The solution of Eq. (7) in units where $e = 1$, $\lambda^k = V^{kl} = 0$ which satisfies Eq. (9), follows:

$$X_\mu(\tau) = X_\mu(0) + P_\mu\tau , \qquad \psi_\mu^k = \prod_{i=1}^{k} \gamma_5^i \gamma_\mu^k \sqrt{\frac{\hbar}{2}} . \tag{10}$$

Superposing these relationships on the physical state vector, we get the mass shell condition expressed in the Bargman-Wigner formalism

$$P_\mu^2 | \Phi > = 0 , \tag{11}$$

the Dirac equation

$$\prod_{i=1}^{k} \gamma_5^i \gamma_\mu^k P_\mu | \Phi > = 0, \quad k = 1, \dots n, \tag{12}$$

and the symmetry condition on function $|\Phi > \alpha_1,...,\alpha_n$ over all indices which eliminates a representation with the maximum spin $n/2$

$$\prod_{i=1}^{k} \gamma_5^i \gamma_\mu^k \prod_{j=1}^{l} \gamma_5^j \gamma_\mu^l | \Phi > = 0. \tag{13}$$

Similarly, we consider motion of a point mass with arbitrary spin. In this case, we expand the phase space, introducing a new anti-commuting coordinate ψ_5^k which corresponds to introducing the "Goldstone field" with the supersymmetrical transformation law $\delta\psi_5^k = m\alpha^k$.[8] For the action to be invariant, we add the following term to the Lagrangian:

$$\delta L = - \frac{e}{2} m^2 - \frac{i}{2} \psi_5^k \dot{\psi}_5^k - \frac{i}{2} m \lambda^k \psi_5^k + i\psi_5^k \psi_5^l V^{kl}. \tag{14}$$

Solutions of the equation of motion which follow from Eq. (14), have the following form:

$$\psi_\mu^k = \sqrt{\frac{\hbar}{2}} \prod_{i=1}^{k} \gamma_5^i \gamma_\mu^k, \quad \psi_5^k = \sqrt{\frac{\hbar}{2}} \prod_{i=1}^{k} \gamma_5^i.$$

Relations of the first kind $P_\mu^2 + m^2 \approx 0, \psi_\mu^k P_\mu + m\psi_5^k \approx 0$ and $\psi_\mu^k \psi_\mu^l + \psi_5^k \psi_5^l \approx 0$ lead to the mass shell condition and Dirac's equation for a fully-symmetric Bargman-Wigner wave function, which describes a particle with mass m and spin $n/2$.

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[1)]Here $\mu = 0,1,2,3$; k, $l = 1,2,...,n$; N, a are space-time indices.

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