Instanton-type solutions in super-symmetrical chiral models

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Recent works[1-5] deal with the study of two-dimensional chiral models both in the pseudo-Euclidian[1,2] and Euclidean[3-5] cases which generalize the **n**-field model.[6] In this article we shall apply a number of results[3-5] to the case of super-symmetrical generalization of these models.[1)]

It should be remembered that in the conventional Euclidean chiral theory the field $\phi(x)(x=(x_1,x_2)\in R^2)$ assumes a value in the nonlinear homogeneous manifold in the ϕ orbit of an associated representation of a simple compact Lie group G: $\Phi = G/H$, where H is a stationary subgroup of the orbit point. Moreover, orbit Φ may be considered as a manifold that is naturally embedded in the algebra $\mathscr{G}$ of the Lie group G. The embedding is given by fixing polynomials that are invariant with respect to the associated representation

$$P_{k_j}(\phi) = c_j, \qquad j = 1, \ldots, n, \tag{1}$$

where n is the rank of group G.

In proceeding to the super-symmetrical case it is convenient to use the terminology of super-space in which each point has conventional coordinates x_1, x_2 and additional anti-commutative coordinates θ_α ($\alpha = 1,2$). We shall assume $\theta_2 = \theta_1^*$, where the asterisk denotes involution in the Grassman algebra (see Ref. 10 for the Grassman algebra).

The involute relation of θ_1 and θ_2 agrees in the Euclidean case with invariance with respect to rotation; moreover, generally speaking, the two different formulations of the problem coincide (solutions that are independent of time in the 2 + 1 theory and solutions in the 1 + 1 theory after a Wick rotation).

We shall determine the Dirac γ-matrices and spinors Ψ and $\bar{\Psi}$ in the following way:

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\Psi = \begin{pmatrix} \psi \\ \psi^+ \end{pmatrix}, \quad \bar{\Psi} = \Psi^+ \gamma^5, \quad \Theta = \begin{pmatrix} \theta_1 \\ \theta_1^* \end{pmatrix} = \begin{pmatrix} \theta \\ \theta^* \end{pmatrix}, \quad \bar{\Theta} = (\theta^*, -\theta). \tag{2}$$

We shall refer to $\hat{\phi}$ as the chiral superpole which corresponds to a field defined on a super-space and which assumes a value in the Grassman algebra; moreover, this field satisfies conditions of the type of Eq. (1):

$$P_{k_j}(\hat{\phi}) = c_j, \qquad j = 1, \ldots n. \tag{1'}$$

However, c is a numerical component ϕ of the field $\hat{\phi}$ and it coincides with the conventional chiral field.

In the case of a real superpole $\hat{\phi}\,(x_1, x_2, \theta, \theta^*)$ the following expansion takes place:

$$\hat{\phi} = \phi + \bar{\Theta}\Psi + \frac{1}{2}\bar{\Theta}\Theta F = \phi + \theta^* \psi - \theta\psi^+ + \theta^*\theta F, \tag{3}$$

where the fields $\phi = \phi^+$ and $F = F^+$ belong to ;the even and ψ, ψ^+ the odd components of the Grassman algebra.

Four independent displacements with respect to x_1, x_2, θ and θ^* take place in the super-space. The displacement generators may be used to construct super-transformation generators

$$E = \frac{\partial}{\partial\theta} - i\theta\partial, \quad \bar{E} = -\frac{\partial}{\partial\theta^*} + i\theta^*\bar{\partial}, \tag{4}$$

where

$$\partial = \partial/\partial z = \tfrac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \partial/\partial\bar{z} = \tfrac{1}{2}(\partial_1 + i\partial_2), \quad z = x_1 + ix_2 \tag{5}$$

and differentiation operators that are anti-commutative with E and $\bar{E}$:

$$D = \frac{\partial}{\partial\theta} + i\theta\partial, \quad \bar{D} = -\frac{\partial}{\partial\theta^*} - i\theta^*\bar{\partial}. \tag{6}$$

We should note that E and $\bar{E}$ correspondingly with D and $\bar{D}$ are "conjugated" with respect to the form

$$< A, B > = \int d^2\theta d^2x \ A^* \cdot B, \tag{7}$$

i.e., for example

$$< A, EB > = (-1)^{\deg A} < \bar{E}A, B > .$$

We shall assume that the field $\hat{\phi}$ converges sufficiently rapidly to the limit $\hat{\phi}_0$ as $|z| \to \infty$. The value of Q (topological charge analog) for such a field may be determined from the following formula:

$$Q = c^{-1}\int d^2\theta\, d^2x (\hat{\phi}, [\bar{D}\hat{\phi}, D\hat{\phi}]_+), \tag{8}$$

where (,) is the scalar product in the Lie algebra $\mathscr{G}$. Q assumes a value on an even component of the Grassman algebra while c, the numerical part of Eq. (8), coincides with the conventional definition of a topological charge.[3] This may readily be seen if Eq. (8) is expressed as a sum of components and the appropriate integration is carried out with respect to θ and θ^*.

$$Q = c^{-1}\int d^2x \{ i\epsilon^{\mu\nu}(\phi, [\partial_\mu\phi, \partial_\nu\phi]) + 3(\Psi\gamma^5\Psi, F) + i([\phi, \bar{\Psi}], \gamma^\mu\partial_\mu\Psi) \\ + 2i(\bar{\Psi}\gamma^\mu\Psi, \partial_\mu\phi) \}, \quad \mu, \nu = 1, 2 . \tag{9}$$

The work (energy) functional for the chiral field may readily be generalized for the super-symmetrical case.[3]

$$S = \frac{1}{2}\int d^2\theta d^2x \{ (D\hat{\phi}, D\hat{\phi}) - ([\hat{\phi}, \bar{D}\hat{\phi}], [\hat{\phi}, D\hat{\phi}]) \} . \tag{10}$$

In our case "duality equations" play an important role as is the case in the chiral theory

$$D\hat{\phi} = [\hat{\phi}, D\hat{\phi}], \qquad \bar{D}\hat{\phi} = -[\hat{\phi}, \bar{D}\hat{\phi}] . \tag{11}$$

It may readily be shown that the "equations of motion" (Euler's equations) which follow from the condition $\delta S = 0$ and allow for the conditions in Eq. (1′), are satisfied when the conditions in Eq. (11) are also satisfied. It can be also shown that as in the case of the conventional chiral theory, in the super-symmetrical case a consequence of the duality equations is

$$S = c\,Q, \tag{12}$$

moreover, c, the numerical part, yields a precise lower bound value for the work S, and the Grassman part is equal to the value of work at a fixed point.

In conclusions, we shall examine in more detail the super-symmetrical generalization of the chiral theory where the field $\phi(x)$ takes on values in the complex projected space: $\Phi = CP^n$.[2,4] In this case, the superpole $\hat{\phi}$ may be expressed in terms of a $(n+1)$-dimensional vector $\hat{\mathbf{u}} = (\hat{u}^{(1)},\ldots,\hat{u}^{(n+1)})$

$$\hat{\phi}_{\alpha\beta} = \frac{1}{n+1}\,\delta_{\alpha\beta} - \hat{u}_\alpha \hat{\bar{u}}_\beta, \qquad \alpha, \beta = 1, \ldots, \quad n+1, \tag{13}$$

where

$$\hat{u}_\alpha = u_\alpha + \theta^* \psi_\alpha - \theta\,\chi^*_\alpha + \theta^*\theta F_\alpha \tag{14}$$

and it satisfies the following condition

$$(\hat{\bar{\mathbf{u}}}, \hat{\mathbf{u}}) = \Sigma\, \hat{\bar{u}}_\alpha \hat{u}_\alpha = 1, \tag{15}$$

while the fields u_α, ψ_α, χ_α and F_α satisfy additional algebraic conditions

$$(\bar{\mathbf{u}}, \mathbf{u}) = 1, \quad (\vec{\chi}\mathbf{u}) + (\bar{\mathbf{u}}\vec{\psi}) = 0, \quad (\bar{\mathbf{u}}\,\mathbf{F}) + (\bar{\mathbf{F}}\,\mathbf{u}) - (\vec{\psi}^*\vec{\psi}) + (\vec{\chi}\vec{\chi}^*) = 0. \tag{16}$$

In this case expressions for S and Q are as follows:

$$S = \int d^2\theta d^2x \{ (\bar{D}\hat{\bar{\mathbf{u}}}, D\hat{\mathbf{u}}) + (\bar{D}\hat{\mathbf{u}}, D\hat{\bar{\mathbf{u}}}) + (\hat{\bar{\mathbf{u}}}, \bar{D}\hat{\mathbf{u}})(\hat{\bar{\mathbf{u}}}, D\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \bar{D}\hat{\bar{\mathbf{u}}})(\hat{\mathbf{u}}, D\hat{\bar{\mathbf{u}}}) \} \tag{17}$$

$$Q = c^{-1}\int d^2\theta d^2x \{ (\hat{\bar{\mathbf{u}}}, \bar{D}\hat{\mathbf{u}})(\hat{\mathbf{u}}, D\hat{\bar{\mathbf{u}}}) - (\hat{\bar{\mathbf{u}}}, D\hat{\mathbf{u}})(\hat{\mathbf{u}}, \bar{D}\hat{\bar{\mathbf{u}}}) - (\bar{D}\hat{\mathbf{u}}, D\hat{\bar{\mathbf{u}}}) - (D\hat{\mathbf{u}}, \bar{D}\hat{\bar{\mathbf{u}}}) \}, \tag{18}$$

and the duality equations

$$D\hat{\mathbf{u}} = \hat{\mathbf{u}}(\hat{\bar{\mathbf{u}}}\, D\,\hat{\mathbf{u}}), \qquad \bar{D}\hat{\mathbf{u}} = -\,\hat{\mathbf{u}}(\hat{\bar{\mathbf{u}}}\,\bar{D}\,\hat{\mathbf{u}}). \tag{19}$$

The use of inhomogeneous coordinates

$$\hat{w}^j = \hat{u}^j\cdot(\hat{u}^{n+1})^{-1}, \qquad j = 1, \ldots, n \tag{20}$$

converts the duality equations to the following

$$\bar{D}\,\hat{w}^j = 0 \qquad (\text{or}\quad D\,\hat{w}^j = 0), \qquad j = 1,\dots, n. \tag{21}$$

The solution of these equations is

$$\hat{w}^j(x_1, x_2, \theta, \theta^*) = w^j(z) + \theta\,\tilde{w}^j(z), \tag{22}$$

where $w^j(z)$ and $\tilde{w}^j(z)$ are analytical functions of z over the entire plane z, including the infinitely-distant point and, therefore, are rational functions; moreover, w^j assumes values on the even and $\tilde{w}^j$ on the odd components of the Grassman algebra.

It may be shown (having introduced appropriate coordinates) that a similar assertion also holds for the case where the field $\hat{\phi}$ is such that c, its numerical part, takes on values in the orbit of an associated representation of an arbitrary compact Lie group, and in particular, for the case of complex Grassman manifolds $\hat{\mathbf{u}}$ must be represented by a $N \times k$ matrix $\hat{U} + \hat{U} = I$.

[1)]The super-symmetrical generalization of a conventional n-field is treated in Refs. 7 and 8; for the pseudo-Euclidean case, the super-symmetrical generalization of chiral models was studied in Ref. 9.

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