A universal generating function for CGC's

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(Submitted 5 March 1979)

Pis'ma Zh. Eksp. Teor. Fiz. **29**, No. 9, 596–598 (5 May 1979)

A simple exponential function is formulated which is expanded into a series with the Clebsch-Gordan coefficients (CGC). The series explicitly identifies all the CGC symmetries.

PACS numbers: 02.30.Gp, 02.30.Lt

Clebsh–Gordan coefficients (CGC) are normally considered as summation coefficients in the 0(3) group in a form given by the angular momenta J, J_1 and J_2. Actually, CGC's are characterized by a higher symmetry that is associated with a permutation group of the three momenta or with the 0(3) group in a three-dimensional space whose coordinates fix the particle numbers. Associated with this group is the nontrivial CGC symmetry and the group matrix elements determine the recurrence formulas for CGC.

Within the framework of this symmetry the angular momenta cease to be fixed numbers and the fomulas of the conventional CGC theory may be summed up in terms of values of the momenta.

A good example of this kind is the generating function for the CGC's which explicitly identifies all the symmetries of these coefficients.

The formula constitutes an expansion of the exponential in terms of the irreducible representations of the symmetry group which may be regarded as the simplest analog of a plane wave expansion in spherical functions.

Clebsh–Gordon coefficients for all the momenta and their projections are defined by the following exponential function:

$$\exp\,(uv + \bar{x}\bar{y}) = \exp\,(1 + xy)\,(uv)$$

$$= \Sigma\,(-1)^{J_1 + J_2 - J}\left[\frac{(J_1 + J_2 + J + 1)\,!}{(2J + 1)(J_1 + J_2 - J)\,!}\right]^{1/2}$$

$$\langle JM/J_1M_1J_2M_2\rangle\, U^{J_1}_{M_1} V^{J_2}_{M_2} X^{J}_{M} \bar{Y}^{J}_{J_1 - J_2}. \qquad (1)$$

The expression is summed over all J and M common with the CGC's. x,y,u and v are all two-component spinors with a scalar product

$$xy = x^1y^2 - x^2y^1 = x^1y_1 + x^2y_2. \qquad (2)$$

Capital letters designate the following monomials

$$A^J_M = \frac{(a_1)^{J-M}(a_2)^{J+M}}{[(J+M)!\,(J-M)!]^{1/2}} \tag{3}$$

and, similarly, for a contravariant monomial.

We note that

$$A^{JM} = (-1)^{\cdot\ M} A^J_{-M} \tag{4}$$

and the scalar product of two monomials is defined by the following formula

$$X^J Y^J = \sum_M X^{JM} Y_{JM}. \tag{5}$$

Components of a spinor with a bar are as follows

$$\bar{x}^1 = v_1 x^1 + v_2 x^2,$$

$$\bar{x}^2 = u_1 x^1 + u_2 x^2 \tag{6}$$

moreover,

$$(\bar{x}\bar{y}) = (xy)(vu). \tag{7}$$

Derivation of Eq. (1) is based on the known expansion[1]

$$\frac{(u_1v_2-u_2v_1)^{J_1+J_2-J}}{(J_1+J_2-J)!}\ \frac{(v_1x^1+v_2x^2)^{J-\mu}(u_1x^1+u_2x_2)^{J+\mu}}{[(J-\mu)!\,(J+\mu)!]}$$

$$= \left[\frac{(J+J_1+J_2+1)!}{(J+J_2-J)!\,(2J+1)}\right]^{1/2} \sum_{M_1M_2} \langle JM|J_1M_1J_2M_2\rangle\, U^{J_1}_{M_1} V^{J_2}_{M_2} \bar{X}^{JM} \tag{8}$$

$$(\mu = J_1 - J_2,\ M = M_1 + M_2).$$

In order to obtain Eq. (1) we note that the second multiplier in the left-hand side is a monomial.

$$\bar{X}^{J\mu} = \frac{(\bar{x}^1)^{J-\mu}\ (\bar{x}^2)^{J+\mu}}{[(J-\mu)!\,(J+\mu)!]^{1/2}}. \tag{9}$$

Multiplying the right-hand side of Eq. (8) by the monomial and summing up over μ and $J_1 + J_2 - J$ we reach the exponential.[1)]

Equation (1) provides all the properties of the CGC symmetry since both $M_1 + M_2$ and $J_1 - J_2$ figure in this equation as projection indices of two momenta. The same equation yields a simple relation for the d-function. To obtain this relation, pick the following values for the spinor components u,v.

$$u = \begin{pmatrix} \sin \frac{\beta}{2} \\ \cos \frac{\beta}{2} \end{pmatrix} \qquad v = \begin{pmatrix} \cos \frac{\beta}{2} \\ -\sin \frac{\beta}{2} \end{pmatrix}$$

$$uv = -1.$$

Substituting this in Eq. (1) we get the following expression for the right-hand side:

$$\Sigma (-1)^{J - J_1 + M_2} \left[\frac{(J_1 + J_2 + J + 1)!}{(J_1 + J_2 - J)!\,(2J + 1)} \right]^{1/2} < JM \mid J_1 M_1 J_2 M_2 >$$

$$\times \frac{\left(\cos \frac{\beta}{2}\right)^{J_1 + J_2 + M_1 - M_2} \left(\sin \frac{\beta}{2}\right)^{J_1 + J_2 - M_1 + M_2}}{[(J_1 - M_1)!\,(J_1 + M_1)!\,(J_2 - M_2)!\,(J_2 + M_2)!]^{1/2}}. \tag{11}$$

Comparing this with the formula for the d-function[2,3] we get instead of Eq. (1) the following expression:

$$\Sigma \frac{(-1)^{J_1 + J_2 - J}}{(J_1 + J_2 - J)!} d^J_{M, J_1 - J_2} (\beta)\, X^{JM}\, \bar{Y}^J_{J_1 - J_2}. \tag{12}$$

We may sum over the parameter $J_1 + J_2 - J$ and as a result we get the multiplier $\exp(-1) = \exp(uv)$ which is reduced with the corresponding multiplier in the right-hand side of Eq. (1). We arrive at the following formula:

$$\exp (xy) = \Sigma (-1)^{J + M} d^J_{MM'} (\beta)\, X^J_{-M}\, \bar{Y}^J_M. \tag{13}$$

or

$$\exp (xy) = \Sigma d^J_{M, J_1 - J_2} (\beta)\, \bar{X}^{JM}\, \bar{Y}^J_{J_1 - J_2} \tag{14}$$

[1)]This monomial is associated with a representation of the second group with quantum numbers J or $J_1 - J_2$. Namely, the angular momentum difference appears here in the role of a projection.

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