Self-similar rotation of an ideal gas

O. I. Bogoyavlenskiĭ
L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted 29 March 1979)

Pis'ma Zh. Eksp. Teor. Fiz. **29**, No. 10, 622–625 (20 May 1979)

Self-similar solutions, which describe the expansion process of a rotating plasma column, are obtained. A nonstationary model of a large-scale vortex in water and a model for dispersion of a rotating gas column are developed. Exact self-similar solutions with a rotation of an ideal gas are shown.

PACS numbers: 47.30. + s

1. The great importance of self-similar solutions, which use as models the different effects in the gas dynamics, is well known.[1-4] Until now, however, the self-similar solutions with rotation were examined only in the diffusion of a vortex in a viscous, incompressible liquid.[3] In spite of the presence of a large number of effects in which the differential rotation of an ideal gas must be taken into account, the self-similar rotation of gas, judging by the well-known monographs,[1-5] was not studied. This is apparently due to the widespread opinion that the self-similar motion of gas must be one-dimensional. In this paper, we give for the first time the self-similar solutions with rotation of an ideal gas, analyze them, and find some applications.

The equations for adiabatic motion of an ideal gas (with the index for the adiabat $\gamma > 1$) in the cylindrical coordinate system r, ϕ, and z_1 have self-similar solutions of the following type:

$$v = rt^{-1}V(\lambda), \quad w = rt^{-1}\Omega(\lambda), \quad u = z_1 t^{-1}U(\lambda),$$

$$\rho = ar^{-k-3}t^{-s}R(\lambda), \quad p = ar^{-k-1}t^{-s-2}P(\lambda). \tag{1}$$

Here, v, wr^{-1}, and u are the radial, angular, and vertical (along the z_1 axis) gas velocities, respectively, ρ is the density, p is the pressure, $\lambda = r/(bt^{\delta})$ is the self-similar variable, and the constants a and b have the dimensions $[a] = ML^kT^s$ and $[b] = LT^{-\delta}$. The self-similar motion of gas (1) are essentially three-dimensional. In the special case $\Omega = U = 0$ the solution of Eq. (1) is the thoroughly studied,[2-4] one-dimensional self-similar motion of gas. The set of equations for gas dynamics for the solutions of type (1), after substitution of $\tau = \ln\lambda$ and $z = \gamma P/R$, is reduced to the following set of equations:

$$\begin{aligned}
&dV/d\tau = V' = \{z(\kappa - 2V - U) + (V - \delta)(V^2 - V - \Omega^2)\}(z - (V - \delta)^2)^{-1},\\
&z' = z(-(\gamma - 1)V' + 2(1 - \gamma V) - (\gamma - 1)U)(V - \delta)^{-1},\\
&u' = u(1 - u)(V - \delta)^{-1}, \quad \Omega' = \Omega(1 - 2V)(V - \delta)^{-1},\\
&R' = R(-V' + s + (k + 1)V - U)(V - \delta)^{-1}, \quad \kappa = (s + 2 + \delta(k + 1))\gamma^{-1}.
\end{aligned} \tag{2}$$

2. As is well known,[5] the equations for magnetic gas dynamics, after the usual idealization (the infinite gas conductivity and the freezing-in of the vertically directed magnetic field, which is orthogonal to the gas velocity) are reduced to the equations for adiabatic motion of gas with the index $\gamma = 2$. Therefore, let us examine the solutions of Eq. (2) for $\gamma = 2$ and $U = 0$ as the model of a rotating plasma column. In this case, Eq. (2) has the following properties: 1) there exists a first integral $\phi_1 = z(V - \delta)\Omega^{-2}$ with the help of which Eq. (2) is reduced to two equations in the z, V plane; 2) there is a line of singular points:

$$V = 1/2, \ \Omega = \Omega_0, \ U = u_0, \ z = \gamma(\Omega_0^2 + 1/4)a^{-1}, \tag{3}$$

where $u_0 = 0$, $\gamma = 0$, $a = 4(\kappa - 1)(1 - 2\delta)^{-1} > 0$, and $\Omega_0 > 0$ is a parameter. The line (3) at $\beta = (1/2 - \delta)(\kappa - 1) - 1/4 > 0$ is divided by the surface $L = z - (V - \delta)^2 = 0$ into two parts $I_0(L < 0,\ 0 < \Omega_0 < \beta^{1/2})$ and $I_1(L > 0,\ \Omega_0 > \beta^{1/2})$. The singular points in the segment I_0 are unstable (and attractive in the segment I_1). Using the integral ϕ_1, we can show that each singular point of the segment I_0 yields a unique separatrix, which, as $r \to \infty$, enters the attractive singular point $Z(V = z = \Omega = U = 0)$. In the corresponding solutions as $r \to \infty$ we have

$$v \sim w \sim r^{(\delta - 1)/\delta}, \quad \rho \sim r^{(2 - 2\delta - \kappa\gamma)/\delta}, \quad p \sim r^{-\kappa\gamma/\delta}. \tag{4}$$

The separatrices I_0Z at $1/2 < \delta < 1$, $\kappa > 1 - \delta$, and $\beta > 0$ determine everywhere regular solutions in which the gas parameters p, ρ, v, and w approach zero as $r \to \infty$. The asymptotics of these solutions as $r \to 0$ are determined by exact solutions corresponding to the singular points (3):

$$v = r/2t, \quad w = \Omega_0 r/t, \quad u = u_0 z_1/t,$$

$$p = C_1 a^{-1}(\Omega_0^2 + 1/4)\, a b^{-a - \beta_1} r^a t^{-a\delta - \kappa\gamma}, \tag{5}$$

$$\rho = C_1 a b^{-a - \beta_1} r^{a - 2} t^{-a\delta - \kappa\gamma + 2}, \quad \beta_1 = (\kappa\gamma - s - 2)/\delta.$$

In the solutions[5] the gas particles, which move along the logarithmic spirals $\phi = C_0 \ln r + C_2$ and $z_1 = C_3$, leave the rotation axis $r = 0$ at $t = 0$. The self-similar solutions, which correspond to the separatrices I_0Z, describe the spreading of the rotating plasma column at which all the parameters of the gas motion when $t \to \infty$ ($r =$ const) decrease. These solutions can be used in any finite region $0 \leqslant r < C$, $t > 0$.

3. The equations of motion of an ideal incompressible liquid in the theory of shallow water are equivalent to the two-dimensional gas dynamics equations for isentropic motion with $\gamma = 2$.[1] The physical conditions for a large-scale vortex in the ocean are: $p \to p_0$ and $\rho \to \rho_0$ when $r \to \infty$. The solutions obtained above (separatrices I_0Z) for $\gamma = 2$, $\kappa = 0$, and $\delta = 1$ [isentropy condition $\kappa = 2(1 - \delta)$ is satisfied] satisfy these conditons. Among the self-similar solutions (1) in the theory of shallow water there are also solutions with moving raptures.

At $\kappa = 2\delta/\gamma$ and $U = 0$ Eq. (2) has a monotonic function which is analogous to the well-known Sedov's[2] "energy integral":

$$H = R\left[\frac{zV}{\gamma} + (V-\delta)\left(\frac{V^2+\Omega^2}{2} + \frac{z}{\gamma(\gamma-1)}\right)\right] = \text{const}\ \lambda^{k-1}.$$

At $\gamma = 2$ integration of Eq. (2) at the level of the integrals $\phi_1 = C_1$ and $H = 0$ reduces to a simple quadrature system.

4. Let us examine the model of the dispersion of a rotating gas column into a vacuum in the half-space $z_1 \geqslant 0$ (in the plane $z_0 = 0$ the vertical velocity of gas $u = 0$). Such a gas column, is produced when air enters the vacuum system, and also is ejected by a rocket engine. When the index for the adiabat is $\gamma = 3/2$ (which is sufficiently close to $\gamma = 1.4$ for ordinary air) Eq. (2) at $U = 1$ has a first integral $\phi_2 = z(V-\delta)^{1/2}\Omega^{-3/2}$ and a line of singular points (3) at $u_0 = 1$ and $\alpha = 3(\kappa - 2) \times (1 - 2\delta)^{-1} > 0$. Using the integral ϕ_2, we can show that each singular point of the segment $I_0(L<0)$ of line (3) yields a two-dimensional separatrix which enters the singular point Z when $r \to \infty$. The self-similar solutions, which correspond to the separatrices I_0Z for the parameters $1/2 < \delta < 1$, $2/\gamma < \kappa < 2$, $\Omega_0^2 < \beta$, $\beta = (1/2 - \delta)(\kappa - 2) > 1/4$, are a model for the dispersion of a rotating gas column into a vacuum. These solutions when $r \to 0$ have the asympotics (5), $u_0 = 1$ and when $r \to \infty$ they have the asymptotics (4); moreover, $u \sim z_1 r^{-1/\delta} \to 0$. The total energy and mass of the gas column of a unit height in these solutions are finite.

For any $\gamma > 1$, $\delta = 1$, and $\kappa > 2/\gamma$ there is another model of the dispersion of a rotating gas column, which corresponds to stable trajectories of Eq. (2) moving from the singular point $Z_1(z = \Omega = 0$ and $V = U = 1)$ to an attractive singular point Z. These solutions have a vacuum which expands from the rotation axis. We have $p = \rho = w = 0$, $v = v_1$, and $u = z_1/t$ on the inner boundary of the gas. The region of the fastest rotation of gas is propagated along the particles with the velocity $v \approx 2^{1/2} v_1$. At $\delta = 1$ and $\kappa = 0$ there are analogous models for the dispersion of a rotating gas column into the atmosphere, in which $p \to p_0$ and $\rho \to \rho_0$ as $r \to \infty$.

I deeply thank S.I. Anisimov and N.A. Inogamov for useful discussions.

[1]L.D. Landau and E.M. Lifshitz, Mekhanika sploshnoĭ sredy (Mechanics of Continuous Media), Gostekhizdat, 1954.

[2]L.I. Sedov, Metody podobiya i razmernosti v mekhanike (Methods of Similarity and Dimensionality in Mechanics), Nauka, M. 1977.

[3]N.E. Kochin, I.A. Kibel', and N.V. Roze, Teoreticheskaya gidromekhanika (Theoretical Hydromechanics), Fizmatgiz, M., 1963.

[4]Ya.B. Zel'dovich and Yu.P. Raĭer, Fizika udarnykh voln i vysokotemperaturnykh gidromekhanicheskikh yavlenkiĭ (Physics of Shock Waves and High-Temperature Hydromechanic Phenomena), Fizmatgiz, M., 1963.

[5]A.G. Kulikovskiĭ and G.A. Lyubimov, Magnitnaya gidrodinamika (Magnetic Hydrodynamics), Nauka, M., 1963.