Nonlinear plane waves in the massless Yang-Mills theory

G. Z. Baseyan, S. G. Matinyan, and G. K. Savvidi
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A new set of solutions of the Yang-Mills equations, which are nonlinear massive plane waves, is obtained and analyzed.

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1. Recently, it was indicated that the vacuum in the Yang-Mills theory, which was examined in the perturbation theory, is not a true vacuum.[1–4] Apparently, the true vacuum is a coherent state of the gluon field with colorless excitations above it.

It is of crucial importance in this connection to find and analyze the classical solutions of the Yang-Mills equations without the external sources in the Minkowski space, which may prove to be useful for construction and study of the structure of quantum vacuum and asymptotic states of the theory.

2. Let us examine the Yang-Mills field without the external sources, which corresponds, for simplicity, to the SU(2) group in the Minkowski space.

The equations of motion have the form

$$\partial_\mu G^a_{\mu\nu} + g\epsilon^{abc} A^b_\mu G^c_{\mu\nu} = 0, \tag{1}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$$

(the Latin letters correspond to 1, 2, 3 and the Greek letters correspond to 0, 1, 2, 3). We seek a solution of Eq. (1) in the coordinate system in which the Poynting vector becomes zero

$$T_{oj} = G^a_{oi}\, G^a_{ji} = 0, \tag{2}$$

$T_{\mu\nu} = -G^a_{\mu\lambda} G^{a\lambda}_{\nu} + \frac{1}{4} g_{\mu\nu} G^a_{\lambda\rho} G^{a\lambda\rho}$ is the energy-momentum of the field tensor. It is convenient to solve Eqs. (1) and (2) together in the gauge: $A^a_0 = 0$, $\partial_i A^a_i = 0$; then

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$$\epsilon^{abc} A_i^b \dot{A}_i^c = 0, \tag{1a}$$

$$\ddot{A}_i - G_{ji,j}^a + g\epsilon^{abc} A_j^b G_{ji}^c = 0, \tag{1b}$$

$$\dot{A}_i^a G_{ij}^a = 0 \tag{2a}$$

(the point above A_i denotes differentiation with respect to time and $H_{ij,k} \equiv \partial_k G_{ij}$). Using Eq. (1a), we can rewrite condition (2a) in the form

$$\dot{A}_i^a (A_{j,i}^a - A_{i,j}^a) = 0. \tag{2b}$$

A sufficient condition for fulfillment of the relation (2b) is

a) $A_{i,j}^a = 0$, b) $\dot{A}_i^a = 0$, c) $A_{i,j}^a - A_{j,i}^a = 0$.

Let us analyze the case a) when the potential depends only on time in the selected coordinate system [see Eq. (2)]:

$A_i^a = A_i^a(t)$.

Thus, Eq. (1b) has the form:

$$\ddot{A}_i^a - g^2 A_j^a A_j^b A_i^b + g^2 A_i^a A_j^b A_j^b = 0. \tag{3}$$

The general solution of Eq. (3) generally depends on 18 integration constants. Equation (1a), however, represents a case in which three of these constants are equal to zero, so that the solution of $A_i^a(t)$ depends on 15 integration constants.

We want to obtain a solution of Eq. (3) in a nine-parameter form:

$$A_i^a = \frac{O_i^a}{g} f^{(q)}(t), \tag{4}$$

where O_i^a is a constant orthogonal matrix

$$O_i^a O_i^b = \delta^{ab} \tag{4'}$$

[Eq. (4) has no summation over a.]

For $f^{(a)}(t)$ we obtain from Eq. (3) the system

$$\ddot{f}^{(a)} + f^{(a)} (f^2 - f^{(a)2}) = 0, \tag{5}$$

where $f^2 \equiv \Sigma_{a=1}^3 f^{(a)^2}$. We are interested here in the particular solution of Eq. (5) when $f^{(1)} = f^{(2)} = f^{(3)} = f$, i.e., $f(t)$ satisfies the equation $\ddot{f}(t) + 2f^3(t) = 0$.

The solution of this equation has the form

$$f(t) = \left(\frac{2g^2}{3}\right)^{1/4} \mu\, cn \left[\left(\frac{8g^2}{3}\right)^{1/4} \mu(t + t_0);\ \frac{1}{\sqrt{2}}\right], \tag{6}$$

where $cn(x;k)$ is the elliptic cosine of the Jacobi argument x and of the module k, t_0 is the arbitrary time origin, and μ^4 is T_{00} in the coordinate system under consideration. The five-parameter solution, which is given by Eqs. (4), (4'), and (6), is periodic in time with a period $T[(3/8g^2)^{1/2}]^{1/4}(4/\mu)^{1/2}K(1/\sqrt{2})$, where $K(x)$ is the total elliptic integral of the first kind. The field intensities, which correspond to this solution, have the form

$$E_i^a = \frac{O_i^a}{g}\dot{f}, \tag{7}$$

$$H_i^a = \epsilon_{ijk}\epsilon^{abc}\frac{O_j^b O_k^c}{2g}f^2. \tag{8}$$

It can be seen from Eqs. (7) and (8) that

$$H_i^a = g(f/\dot{f})^2\epsilon_{ijk}\epsilon^{abc}E_j^b E_k^c. \tag{9}$$

It can also be seen from Eqs. (4') and (7) that the O_i^a matrix is a polarization matrix of the field intensities; the three E^a vectors are mutually orthogonal in our coordinate system and, as follows from Eq. (9), the H^a vectors are parallel to the E^a vectors.

3. It can easily be seen that the argument of the periodic solution of (6) as a result of the Lorentz transformation $x_\mu = a_\mu^\nu(v)x'_\nu$ becomes $kx = k_\mu x_\mu$, where $k_0 = \mu\gamma$ and $k_i = \mu v_i\gamma[\gamma = (1-v^2)^{-1/2}]$, i.e., $k^2 = \mu^2$, so that μ plays the role of the mass; hence we are dealing with a massive nonlinear plane wave.

For the potential $A_\mu^a(x)$ we obtain:

$$A_\mu^a(kx) = a_\mu^\nu(v)\frac{O_\nu^a}{g}f(k(x+x_0)),\quad O_0^a = 0. \tag{10}$$

The fields E_i^a and H_i^a in Eqs. (7) and (8) can be transformed analogously. Our solutions differ from Coleman's solutions[5] in that the Poynting vector in our case is not equal to the energy density ($k^2 = \mu^2$). The rejection of this condition leads (already at the classical level) to the origination of the mass μ in the nonlinear plane wave (10).

In conclusion, we note that the obtained nonlinear plane wave with the mass produced due to its nonlinearity may prove to be a useful tool for quantization of the nonlinear theory.

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