Stochastization of vortices

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The range of configuration temperatures is determined, in which quasi periodicity for a system of four linear vortices in unlimited space is missing. It is pointed out that stochastization of a smaller number of vortices is possible in the presence of boundaries or an external flow.

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1. For plasma dynamics, superfluidity, and geophysics and also for the general theory of dynamic systems, it is important to know the behavior of linear vortices in an ideal liquid (plasma model in the magnetic field). In the absence of quasi periodicity the system is not fully integrable (the method of the *L-A* pair and the technique of backscattering cannot be used for it) and hence stochastization can be expected.

A three-vortex system in unlimited space (US) can be integrated by quadrature.[1,2] It was shown by Novikov and Sedov[3] that a system of four identical vortices in US has no quasi periodicity in the region of the configuration temperature (5)$\Theta \sim 2.4$.

2. The Cartesian coordinates of the vortices $z_\alpha(t) = x_\alpha(t) + iy_\alpha(t)$ with the intensities κ_α satisfy the Hamilton equations

$$\kappa_\alpha \frac{dz_\alpha}{dt} = -i \frac{\partial H}{\partial \bar{z}_\alpha}, \qquad \bar{z}_\alpha = x_\alpha - iy_\alpha . \tag{1}$$

In the absence of external flow for US

$$H = -\frac{1}{2\pi} \sum_{\alpha < \beta} \kappa_\alpha \kappa_\beta \ln l_{\alpha\beta}, \qquad l_{\alpha\beta} = |z_\alpha - z_\beta| . \tag{2}$$

From the invariance (2) relative to the shifts and rotations we obtain the motion integrals

$$Z = \sum_\alpha \kappa_\alpha z_\alpha, \qquad I = \sum_\alpha \kappa_\alpha z_\alpha \bar{z}_\alpha \tag{3}$$

and their combination[1]

$$M = \sum_{\alpha,\beta} \kappa_\alpha \kappa_\beta l_{\alpha\beta}^2 . \tag{4}$$

In the case of identical vortices ($\kappa_\alpha \equiv \kappa$), the sole determining dimensionless parameter of the microcanonical distribution, which was constructed from the invariants (2) and (4), is the configuration temperature[1]:

$$\Theta = \prod_{\alpha < \beta} \frac{r^2}{l_{\alpha\beta}^2}, \qquad N(N-1)\, r^2 = \sum_{\alpha,\beta} l_{\alpha\beta}^2 \tag{5}$$

(r is the rms distance between N vortices).

We give the characteristic values of Θ for $N = 4$, which correspond to the steady-state configurations. The minimum value of $\Theta_0 = 2^{10}3^{-6} \approx 1.40$ corresponds to a stable rotation of the vortices situated at the vertex of the square. The unstable rotation of three

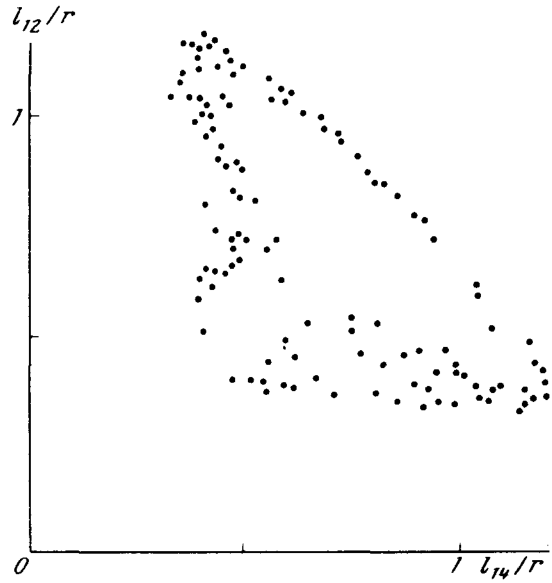

FIG. 1. Poincaré projection at $\Theta \approx 8.71$.

vortices with the period T, which are situated at the vertices of an equilateral triangle, around the fourth vortex located at the center,[3] corresponds to $\Theta_1 = 2^6 3^{-3} \approx 2.37$. Finally, $\Theta_2 = 4\Theta_1$ corresponds to an unstable rotation of four vortices situated on a single line.

3. The numerical experiments were carried out in such a way that the entire permissible region $\Theta \geqslant \Theta_0$ could be examined. The phase trajectories, which were obtained by integrating (1) with the initial data corresponding to different Θ, were observed at large intervals of time. We traced the sequence of by-passing the cells of the breakdown of the phase space,[3] which correspond to different convex and nonconvex

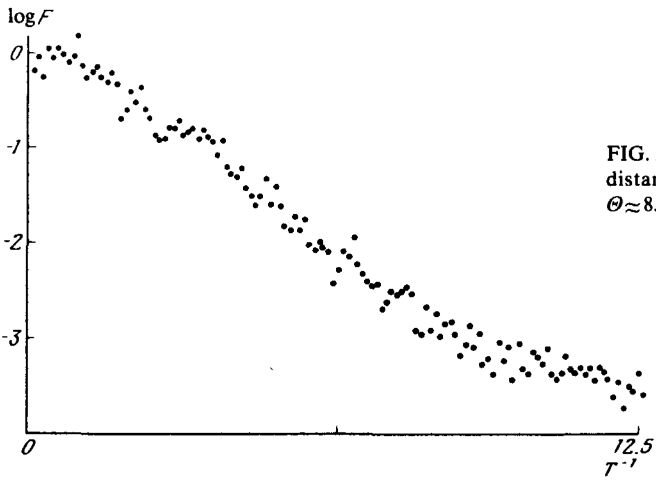

FIG. 2. The F spectrum for one of the distances between the vortices at $\Theta \approx 8.71$.

configurations, and determined the points of intersection of the trajectories and the cell boundaries (Poincaré image) and the spectra of magnitude $z_\alpha(t)$ and $l_{\alpha\beta}(t)$. The numerical experiments showed that at $\Theta \geqslant \Theta_2$ the vortices form clusters (groups) between which there is no exchange of vortices. This indicates that some transitions between the cells are forbidden. The sequence of by-passing the cells was periodic and the points on the Poincaré projection were on the curve. In the range $\Theta_0 < \Theta < \Theta_1$, initially (at $\Theta < \Theta_* \sim 2.17$, Θ_* corresponds to the three vortices on the straight line) the trajectories are located entirely in one of the cells (convex configuration), then (at $\Theta > \Theta_*$) they can enter the four adjacent cells (nonconcave configurations); the other transitions between the cells are forbidden. The by-passing of cells is periodic. In both ranges $\Theta_0 < \Theta < \Theta_1$ and $\Theta > \Theta_2$ the motion, which is quasi periodic with two frequencies, occurs along the surface of a two-dimensional torus. The picture is different in the interval $\Theta_1 < \Theta < \Theta_2$. Here all the allowed transitions between the convex and nonconvex configurations are realized, the cells are by-passed nonperiodically, and the points on the Poincaré projections occupy a two-dimensional region (Fig. 1)-the motion in the phase space is three dimensional. The spectra of the different phase variables do not have peaks (Fig. 2).

4. The absence of quasi periodicity and the three-dimensional motion of the four vortices in US indicate that the system of vortices in general does not have "hidden" motion integrals (those not associated with the symmetry restrictions). In the presence of boundaries and external flow, the loss of quasi periodicity and stochastization can occur with a smaller number of vortices. There should be not less than three independent variables for the relative motion of vortices. Thus, for stochastization generally three vortices are sufficient for a flow above a flat boundary or inside a cylinder. In the case of a cylinder, this can be checked experimentally by a visual inspection of the quantized vortices in liquid helium.[4] If the region is devoid of displacement symmetry and rotational symmetry (for example, regions with a semicircular or rectangular cross section), then we can expect the loss of quasi periodicity for two vortices. One vortex is sufficient for stochastization of the trajectories of the liquid particles in such a region. This also applies to US, if there is an external flow which does not have the indicated symmetry properties.

[1]E.A. Novikov, Zh. Eksp. Teor. Fiz. **68**, 1868 (1975) [Sov. Phys. JETP **41**, 937 (1975)].
[2]A. Hassan, Preprint, Cornell Univ., 1978.
[3]E.A. Novikov and Yu.B. Sedov, Zh. Eksp. Teor. Fiz. **75**, 868 (1978) [Sov. Phys. JETP. **48**, 440 (1978)].
[4]J.A. Williams and R.E. Packard, Phys. Rev. Lett. **33**, 280 (1974).