Existence of heavy particles in gauge field theories

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We formulate the conditions for the existence of heavy particles in gauge theories with arbitrary singly connected compact group G. The case of the active representation of the SU(3) group is investigated in detail.

In this article we consider a system of k scalar fields ϕ_a and Yang-Mills vector fields A^j_μ, described by the Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi_a\nabla_\mu\phi_a - V(\phi)$$

(the fields ϕ_a transform in accordance with the k-dimensional unitary representation of the singly-connected compact group G, while the fields $A_\mu = (A^1_\mu, \ldots, A^m_\mu)$ assume values in the algebra of the Lie group G, $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$, ∇_μ is the covariant derivative, and $V(\phi)$ is a G-invariant function).

In the case when ϕ_a transforms in accordance with the three-dimensional representation of the $SU(2)$ group, $V(\phi) = c(\phi_a\phi_a - F^2)^2$, it was shown in[1] and[2] that in this theory there exist, besides the usual particles, also heavy particles called extremons in[1]. We indicate here the conditions for the existence of these particles for an arbitrary group G, and investigate in detail the case of the octet representation of the $SU(3)$ group. It follows from our condition, in particular, that in a situation where only one vector field does not acquire mass (there is one photon) there exists one extremon (monopole).

We use the name "classical vacuum" for the point of k-dimensional space at which the function $V(\phi)$ has a minimum (this minimum is assumed to be equal to zero); the aggregate of all the classical vacuums is designated by R. We assume that any classical vacuum can be obtained from any other one by the transformation of the group G, i.e., the group G acts on R transitively (this means that the degeneracy of the vacuum is due entirely to the action of the group). The stationary subgroup (i.e., the subgroup containing the transformations that leave a fixed classical vacuum in place) will be designated by H. The case when only one of the vector fields does not acquire mass corresponds to $H = U(1)$. It turns out that the sufficient condition for the existence of extremons is that the subgroup H not be singly connected; the same condition is necessary if we wish the topological considerations to ensure stability of the extremon. If $\pi_1(H)$ (the fundamental group of the H group) has s generators, then there exists s stable extremons (for the definition of the homotopic groups π_i see, e.g.,[3]).

In fact, extremons are defined in[1] as states in which the energy functional $\epsilon(\phi_a, A^j_\mu)$ reaches a local minimum. If $\epsilon(\phi_a, A^j_\mu) < \infty$, then one can assume, without a significant loss of generality, that as $r \to \infty$ we have $\phi_a(\mathbf{r}) \approx \lambda_a(\mathbf{n})$, where $\mathbf{r} = r\mathbf{n}$ and the function $\lambda(\mathbf{n}) = (\lambda_1(\mathbf{n}), \ldots, \lambda_k(\mathbf{n}))$ takes on values in the manifold of the classical vacuums R. The function $\lambda(\mathbf{n})$ is the mapping of the sphere S^2 in R, meaning that it defines an element of the group $\pi_2(R)$. Two states with finite energy can be continuously transformed one into another if and only if they define one and the same element of the group $\pi_2(R)$ (i.e., the space of states with finite energy breaks up into pieces—connectivity components—that are in mutually unique correspondence with the elements of the group $\pi_2(R)$). The minimum of the energy on a given connectivity component may not be reached, however, and the elements of the $\pi_2(R)$ group corresponding to those connectivity components on which the minimum is reached make up the system of generators of the group $\pi_2(R)$. It now remains only to note that by virtue of simple topological considerations we have $\pi_2(R) = \pi_2(G/H) = \pi_1(H)$. If G does not act on R transitively, then the topological condition for the existence of extremons is formulated in a more complicated manner; we note only that in this case $\lambda(\mathbf{n})$ is the mapping of S^2 into one of the orbits, lying in R, of the group G.

We consider by way of example the case when ϕ_a transforms in accordance with the octet representation of the $SU(3)$ group, and $V(\phi)$ is a polynomial of degree $\leqslant 4$ (this choice of $V(\phi)$ ensures renormalizability of the theory). If $V(0) > \min V(\phi)$, then one can prove the existence of one extremon (if $V(\phi) \neq c_0 + c_1(\phi_a\phi_a) + c_2(\phi_a\phi_a)^2$, then $H = U(2)$, $\pi_1(H)$ has one generator, and the arguments advanced above are applicable; in the opposite case we have $R = S^7$ and the group $SU(3)$ does not act transitively on R, so that additional arguments are needed). This extremon can be represented in the form

$$\phi(\mathbf{r}) = \alpha(r)\vec{\lambda}\mathbf{n} + B(r)\lambda_8,$$

$$A_i(\mathbf{r}) = \gamma(r)\epsilon_{ijl}n_j\lambda_l, \quad i = 1, 2, 3, \quad A_4(r) = 0$$

(here $\lambda_1, \ldots, \lambda_8$ are the usual generators of $SU(3)$; $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$). We can easily write down a system of equations for the functions $\alpha(r)$, $\beta(r)$, and $\gamma(r)$; it is possible to prove rigorously that this system has a solution (the rigorous proof of this fact is new also for the $SU(2)$ group). If $V(\phi)$ is a polynomial of degree higher than 4, then the case $H = U(1)\times U(1)$ is possible; then there exists two stable extremons, which can be expressed in analogous fashion.

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