Concerning the existence of monopoles in gauge field theories

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A simple topological criterion is presented for the existence of monopoles in gauge-invariant theories with arbitrary compact symmetry group G.

Recently t'Hooft[1] and Polyakov[2] have obtained static solutions of the classical $SU(2)$ equations-gauge invariant field theories that describe monopoles.

The purpose of this article is to indicate a simple necessary condition for the existence of solutions of this type in gauge-invariant theories with arbitrary compact symmetry group[1)] G. Namely, this condition is that the second homotopic group of the factor-space G/H be nontrivial: $\pi_2(G/H)\neq 0$. Here H is a subgroup in G and depends on the choice of the boundary conditions (as $r\to\infty$) for the solution of interest to us.

Consider a system of interacting vector (Yang-Mills) fields A^j_μ ($\mu=1,2,3, j=1,\ldots,n$, n is the dimensionality of G), which transform in accord with an adjoint representation of the group G and scalar (Higgs) fields ϕ_a ($a=1,\ldots,N$) transforming in accordance with an N-dimensional unitary irreducible[2)] representation $T(g)$.

The Lagrangian of this system is

$$L = -\frac{1}{4}F_{j,\mu\nu}F^{j,\mu\nu} - \frac{1}{2}(D_\mu\phi_a)(D^\mu\phi^a) - U(\phi), \quad (1)$$

where

$$F^j_{\mu\nu} = \partial_\mu A^j_\nu - \partial_\nu A^j_\mu + gC^j_{kl}A^k_\mu A^l_\nu,$$
$$D_\mu\phi_a = \partial_\mu\phi_a + igA^j_\mu(T_j)^b_a\phi_b, \quad (2)$$

C^j_{kl} are the structure constants of the Lie algebra of the group G, and T_j are infinitesimal operators of the representation $T(g)$. It is assumed that (a) the potential $U(\phi)$ is a G-invariant function of the fields ϕ_a, (b) the absolute minimum U_0 of the potential $U(\phi)$ is reached only at finite values $\phi_a=\chi_a\neq 0$.

The equations of motion for the fields ϕ_a and A^j_μ can be easily obtained by varying L with respect to ϕ_a and A^j_μ. For the vacuum solution (i.e., for the solution corresponding to the absolute minimum of the energy) we have $\phi_a(\mathbf{r})\equiv\chi_a$ and $A^j_\mu(\mathbf{r})\equiv 0$. We seek the solution describing the monopole in the class of time-independent functions $\phi_a(\mathbf{r})$ and $A^j_\mu(\mathbf{r})$ with the following asymptotic behavior as $r\to\infty$:

$$\phi_a(\mathbf{r})\sim\hat{\phi}_a(\mathbf{n}), \qquad A^j_\mu(\mathbf{r})\sim\frac{1}{r}\hat{A}^j_\mu(\mathbf{n}), \quad \mathbf{r}=r\mathbf{n}, \quad \mathbf{n}^2=1, \quad (3)$$

and we assume that the fields $\phi_a(\mathbf{n})$ depend significantly on $\mathbf{n}$.[3)] We note that by virtue of the equations of motion the $\hat{A}^j_\mu(\mathbf{n})$ are fully determined by the functions $\hat{\phi}_a(\mathbf{n})$. Following,[2] we choose $\hat{\phi}_a(\mathbf{n})$ in the form

$$\hat{\phi}_a(\mathbf{n}) = \Omega^b_a(\mathbf{n})\,\phi^{(0)}_b, \quad \Omega^b_a(\mathbf{n}) = T^b_a(g(\mathbf{n})), \quad (4)$$

where $\hat{\phi}^{(0)}$ is a fixed vector, such that $U(\hat{\phi}^{(0)})=U_0$.

It is seen from (4) that the vectors $\hat{\phi}(\mathbf{n})$ belong to a definite orbit of the representation $T(g)$, i.e., to a set of vectors of the type $T(g)\phi^{(0)}$, where g runs through the entire group G. It is known that this orbit can be regarded as the factor-space G/H, where H is a stationary subgroup of the vector $\hat{\phi}^{(0)}$, i.e., the set h such that $T(h)\hat{\phi}^{(0)}=\hat{\phi}^{(0)}$.

Thus, the boundary conditions postulated by the functions $\hat{\phi}_a(\mathbf{n})$ define the mapping of the two-dimensional sphere $S^2=\{\mathbf{n}:\mathbf{n}^2=1\}$ on the orbit G/H, $\hat{\phi}: S^2 \to G/H$. The mapping $\hat{\phi}'$, which can be continuously deformed into $\hat{\phi}$, will be called homotopic to $\hat{\phi}$.[4)] The set of classes of mappings that are homotopic to one another constitutes the group $\pi_2(G/H)$. Since a change to another gauge yields a mapping of $\hat{\phi}$ that is homotopic to $\hat{\phi}$, it follows that nontriviality of the group $\pi_2(G/H)$ is a necessary condition for the existence of monopoles.

We present the results of the calculations of $\pi_2(G/H)$ in the cases of greatest interest: (a) singly-connected group: $\pi_2(G/H)=\pi_1(H)$, $\pi_1(G/H)=0$; (b) $G=\tilde{G}/C$, where $\tilde{G}$ is a singly-connected group with a finite center; C is a subgroup of the center of $\tilde{G}$:

$$\pi_2(G/H)=\pi_2(\tilde{G}/H)=\pi_1(H), \qquad \pi_1(G/H)=C.$$

Let us illustrate these general formulas with the following physical examples:

1. Let $G=SU(2)$, $T(g)=T^I(g)$, with I the isotopic spin. (a) I is half-integer; the stationary group of any vector $\hat{\phi}^{(0)}$ consists of only one element. Consequently $\pi_1(H)=0$ and there are no solutions of the monopole type. (b) I is integer; for a vector $\hat{\phi}^{(0)}$ with zero isospin projection on the t_3 axos. $H=U(1)$, $\pi_1(H)=Z$ (Z is the group of integers), and the possible solutions of the monopole type are characterized by a single integer.

2. Let $G=SU(3)$ and $T(g)=T^{(p,q)}$, where p and q are nonnegative integers. (a) for the representations $T^{(1,0)}$ of dimensionality 3, at any choice of the nonzero vector $\hat{\phi}^{(0)}$, we have $H=SU(2)$ and $\pi_1(H)=0$, and there are no solutions of the monopole type. (b) Let $T^{(1,1)}$ be the associate representation $\dim T^{(1,1)}=8$. The vector $\hat{\phi}^{(0)}$ takes in this case the form of a Hermitian 3×3 matrix, and $\mathrm{Tr}\,\phi^{(0)}=0$. If all three eigenvalues of $\hat{\phi}^{(0)}$ are different, then $H=U(1)\times U(1)$ and $\pi_1(H)=Z+Z$. Solutions of this kind are numbered by two integers. On the other hand, if the two eigenvalues coincide, then $H=U(2)$ and $\pi_1(H)=Z$, i.e., these solutions are numbered by a single integer.

3. In the Weinberg model $G=SU(2)\times U(1)$ and $H=U(1)$ and is imbedded in G irregularly. In this case $\pi_2(G/H)=0$ and there are no solutions of the monopole type.

We note in conclusion that physical interest attaches only to stable solutions with finite energy. To find these solutions, additional investigations are needed.

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1) After completing the work, the authors learned that certain results in this direction were obtained by A. S. Shvarts.

2) The reducible representations of the G group are also of interest. They can be treated by the same method.

3) Solutions of this type, but only for Yang-Mills Fields and G-$SU(2)$, were considered earlier by Yang and Wu.[3]

4) All the topological concepts used in the text are contained in[4].

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