Contribution to the theory of the spectrum of a Bose system with condensate at small momenta

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(Submitted September 26, 1974)
ZhETF Pis. Red. **21**, No. 1, 3–6 (January 5, 1975)

We obtain a result $\Sigma_{02}(0)=0$ that eliminates the divergences in the derivation of a formula for the Green's function of a Bose system with a condensate at small momenta. A simple convergent diagram expression is obtained for $1/c^2$ (c is the speed of sound). The conditions for the applicability of calculations using a small parameter is discussed.

Gavoret and Nozieres[1] obtained a number of exact relations for a Bose system with condensate, in the limit of small momenta at $T=0$, and in particular

$$G'(p)=-\hat{G}(p)=\frac{n_0 mc^2}{n(\epsilon^2-c^2p^2+i\delta)},\qquad c^2=\frac{n}{m}\frac{d\mu}{dn},\qquad p\equiv(\mathbf{p},\epsilon)\to 0. \tag{1}$$

However, they themselves call attention to a definite shortcoming of their analysis, namely, the intermediate steps contain expressions that diverge as $p\to 0$, and these expressions combine into quantities with microscopic meaning only in the last stage. To avoid divergences, they had to introduce into the spectrum an unphysical gap Δ, which is assumed to tend to zero in the final expression; the procedure of taking the limit as $p\to 0$ is in this case not justified until Δ vanishes. The derivation of (1) in[1] is unsatisfactory also in another respect, in that it uses essentially the assumption $\Sigma_{02}(0)\neq 0$, which, as we shall show, is incorrect.

In this article we obtain as an exact result

$$\Sigma_{02}(0)=0 \tag{2}$$

and propose for formulas (1) a derivation that takes (2) into account and contains no divergences whatever in the intermediate stages of the calculations. A simple diagram expression is obtained in this case for $1/c^2$, and its convergence results in a microscopic verification of the inequality $c\neq 0$ (independent of the microscopic requirement that the compressibility be finite). We consider here also the conditions under which it is permissible, for a Bose system with an arbitrary small parameter α, to discard diagrams of higher orders in α that diverge as $p\to 0$.

We write down the exact equation for $\Sigma_{ik}(p)$, separating the term that diverges in the approximations (Fig. 1; $\tilde{\Sigma}(p)$ is irreducible relative to two particle lines). Recognizing that $G'(p)=-\hat{G}(p)=G'(-p)=\hat{G}(-p)(p\to 0)$ we find

$$\gamma(0,0)=\frac{1}{2}\Big[\Gamma(0,0,0)-\Gamma_1(0,0,0)\Big]=\frac{1}{n_0}\Sigma_{02}(0). \tag{3}$$

At $p=0$, the integrand in the term with γ (Fig. 1) takes at small q the form

$$i\frac{n_0^2m^2c^4}{n^2(\omega^2-c^2q^2+i\delta)^2}\left[\frac{1}{n_0}\Sigma_{02}(0)+\tilde{\gamma}(0,q)\right],\qquad \tilde{\gamma}(0,0)=0, \tag{4}$$

so that we obtain for $\Sigma_{02}(0)$ an exact self-consistent equation, the only finite solution of which is (2). The Hugenholtz-Pines relation[2] thus takes the form $\mu=\Sigma_{11}(0)$.

We note that (2) does not mean $c=0$, as might seem from a comparison of relations (1) and (5) of[3]

$$G'=-\hat{G}=\Sigma_{02}(0)/B(\epsilon^2-c^2p^2+i\delta) \tag{5}$$

since $B=0$, i.e., (5) contains an uncertainty. Indeed, using the equations of[1]

$$\Sigma_{02}(0)=n_0\left(\frac{\partial^2E'}{\partial n_0^2}\right)_\mu,\qquad \frac{\partial\Sigma_{11}(0)}{\partial\epsilon}=-\left(\frac{\partial n'}{\partial n_0}\right)_\mu \tag{6}$$

we obtain

$$\frac{d}{d\mu}[\Sigma_{11}(0)-\Sigma_{02}(0)]=1=\frac{\partial^2E'}{\partial n_0\partial\mu}+\frac{\partial^2E'}{\partial n_0^2}\frac{dn_0}{d\mu}=-\frac{\partial\Sigma_{11}(0)}{\partial\epsilon}+\frac{1}{n_0}\Sigma_{02}(0)\frac{dn_0}{d\mu}.$$

(the direct derivatives correspond to the physical changes of the parameters), i.e., with allowance for (2) we have

$$\partial\Sigma_{11}(0)/\partial\epsilon=1. \tag{7}$$

Substitution of (2) in (7) in the definition of B[3] yields $B=0$.

To obtain the limiting form of the Green's function G', $\hat{G}(p\to 0)$, with allowance for (2), it is important to point out the character of the nonanalytic terms $\Delta\Sigma_{ik}$ in the expansion of $\Sigma_{ik}(p)$ near $p=0$, which are due to the diagram with $\gamma(p,q)$ (Fig. 1). It is important that in the lowest order in p the nonanalytic corrections to

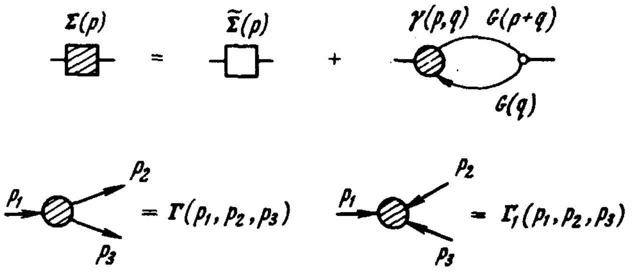

FIG. 1.

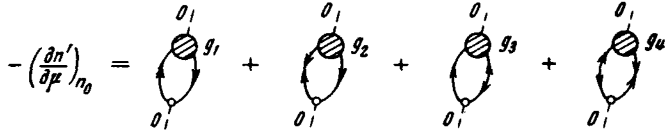

FIG. 2.

$\Sigma_{11}(p)$ and $\Sigma_{02}(p)$ coincide, $\Delta\Sigma_{11}=\Delta\Sigma_{02}\equiv\Delta\Sigma(p)$. Recognizing that $\gamma(0,0)=0$, we obtain $\Delta\Sigma(\epsilon^2=\epsilon_p^2)\sim p^2\ln p$, $\Delta\Sigma(\epsilon^2\approx\epsilon_p^2)\sim p(\epsilon\pm\epsilon_p)\ln(\epsilon\pm\epsilon_p)$ [for the "bare" vertex $\gamma^{(0)}(p,q)$, the contribution would be respectively $\sim\ln p$ and $\sim p^{-1}(\epsilon\pm\epsilon_p)\ln(\epsilon\pm\epsilon_p)$]. Substituting the expansions

$$\Sigma_{11}(p)=\mu+\epsilon+\Delta\Sigma(p)+a\epsilon^2+bp^2+\dots$$

$$\Sigma_{02}(p)=\Delta\Sigma(p)+a_1\epsilon^2+b_1p^2+\dots$$

in the exact expressions for G' and $\hat{G}$ in terms of Σ_{ik}[3] [see (2), (7)] and taking into account the fact that $\Delta\Sigma(p)\gg\epsilon^2$, p^2 as well as[1]

$$a-a_1=-\frac{1}{2n_0}\left(\frac{\partial n'}{\partial\mu}\right)_{n_0},\quad b-b_1=\frac{n'}{n_0}\frac{1}{2m},$$

we obtain (1), with

$$c^2=n/m\left(\frac{\partial n'}{\partial\mu}\right)_{n_0} \tag{8}$$

Since $(\partial n'/\partial n_0)_\mu=-1$ according to (6) and (7), we have $dn/d\mu=(\partial n'/\partial\mu)_{n0}$ and Eq. (8) agrees with (1). It is remarkable that the static susceptibility $F_{44}(\epsilon=0,\ p\to 0)=dn/d\mu$ coincides with the analogous characteristic of the system of supercondensate particles at a fixed number of particles in the condensate.

The convergence of the skeleton diagrams for $(\partial n'/\partial\mu)_{n0}$ (Fig. 2) as $q\to 0$ follows from the equality

FIG. 3.

$$g_1-g_2-g_3+g_4=2\left\{1-\frac{\partial}{\partial\mu}\left[\overset{\circ}{\Sigma}_{11}(0)-\Sigma_{02}(0)\right]\right\}=\frac{2}{n_0}\Sigma_{02}(0)\frac{dn_0}{d\mu}=0$$

(see (1), (6), and (2)).

In conclusion, we examine why the result (2) is not obtained approximately for models with a small parameter α, for example for a rarefied system in the ladder approximation (Fig. 3a). As $p\to 0$, many diagrams made up of Bogolyubov or exact Green's functions diverge, starting with the simplest ones, and the most rapidly diverging are diagrams with three-pronged vertices, for example for $\Sigma_{ik}(p)\sim 1/p^{n-2}$ (Fig. 3b). It can be shown that the degree of divergence m is proportional to a power of the small parameter l ($l/m\sim 1$), so that a cancellation of the small parameter takes place in a narrow range of momenta $p_1\sim\alpha^{l/m}p_0\ll p_0$ (p_0 is the characteristic momentum of the interaction). The substitution $\Gamma\to\Gamma^0$ in skeleton diagrams is permissible if the resultant diagram converges (for example, F_{33}[4]), but is not valid in the case of a divergence, even if one takes some convergent sequence of diverging diagrams (Σ_{02}, F_{44}), meaning that the region $p\to p_1$ remains essential; it is necessary here to use the exact connection with the quantities for which the diagrams with $\Gamma^{(0)}$ converge (of the type F_{44} with F_{33}[4]).

[1]J. Gavoret and P. Nozieres, Ann. of Phys. 28, 349 (1964).
[2]N. M. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).
[3]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody Kvantovoi teorii polya v statisticheskoi fizike (Quantum Field Theoretical Methods in Statistical Physics), Moscow, 1962 [Pergamon, 1965].
[4]S-K. Ma, H. Gould, and V. K. Wong, Phys. Rev. A3, 1453 (1971).