Power-law solutions of the Boltzmann kinetic equation, describing the spectral distribution of particles with fluxes

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We obtain exact power-law solutions that cause the collision integral of the Boltzmann kinetic equation to vanish.

Power-law distributions (spectra) are encountered quite frequently, with cosmic rays as an example.[1] The question is, how are these spectra produced. Thus, for example, they can result from the interaction of particles with waves in a turbulent plasma,[2] etc.

We show in this paper that power-law distributions can result from direct interaction between particles. This class of solutions of the Boltzmann kinetic equation presupposes the existence of an energy source and an energy sink, and is analogous to the solutions first obtained by Zakharov in the theory of weak turbulence for waves.[3] The properties of systems with such distribution functions should differ greatly from those of equilibrium systems.

Let us find a solution that causes the Boltzmann collision integral to vanish:

$$I_{col}\{n\} = \int d\tau_p W_{pp_1|p_2p_3} f_{pp_1|p_2p_3}, \qquad (1)$$

$$W_{pp_1|p_2p_3} = U_{pp_1|p_2p_3}\,\delta(\mathbf{p}+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\,\delta(E+E_1-E_2-E_3), \qquad (2)$$

$$f_{pp_1|p_2p_3} = n(\mathbf{p}_2)n(\mathbf{p}_3) - n(\mathbf{p})n(\mathbf{p}_1), \quad d\tau_p = d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3$$
$$E(\mathbf{p}) = E, \quad E(\mathbf{p}_1) = E_1 \text{ etc.} \qquad (3)$$

Assuming that the self-similarity and isotropy conditions

$$E(\lambda\mathbf{p}) = \lambda^{\beta}E(\mathbf{p}), \quad U_{\lambda p\,\lambda p_1|\lambda p_2\,\lambda p_3} = \lambda^{m} V_{pp_1|p_2p_3}, \qquad (4)$$

$$E(\hat{g}\mathbf{p}) = E(\mathbf{p}), \quad U_{\hat{g}p\,\hat{g}p_1|\hat{g}p_2\,\hat{g}p_3} = U_{pp_1|p_2p_3} \qquad (5)$$

are satisfied ($\hat{g}$ is the rotation operation, and the particles can be of different types but must have the same homogeneity exponent β), and using the conformal symmetry transformations (see[3,4] and the review[5]) previously applied to the kinetic equations for waves in weak-turbulence theory, we obtain for isotropic solutions $n = E^s$ (in the case of particles of only one type);

$$I_{col}\{n\} = \frac{E^{\nu}}{4}\int d\tau_p W_{pp_1|p_2p_3}\{E^{-\nu} + E_1^{-\nu} - E_2^{-\nu} - E_3^{-\nu}\} \qquad (6)$$

$$\nu = 2s - 1 + \frac{m + 3d}{\beta} \qquad (7)$$

(d is the dimensionality of p-space). As seen from (6) and (2), there are two solutions

$n = E^{s_1}(\nu = -1)$ and $= E^{s_0}(\nu = 0)$ of the equation $I_{col}\{n\} = 0$.

$$s_1 = -\frac{m + 3d}{2\beta}, \qquad s_0 = s_1 + \frac{1}{2} \qquad (8)$$

describing the distributions with constant energy flux and with constant particle-number flux over the spectrum.

For particles interacting in accordance with the power law $V(r) = r^{-\alpha}$ (Coulomb forces, Van der Waals forces, etc.), the Born matrix element in the Boltzmann equation depends on the momentum transfer $\hbar\mathbf{k} = \mathbf{p}_1 - \mathbf{p}_3$

$$U_{\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3} = \frac{2\pi}{\hbar} |V(k)|^2, \qquad V(k) = \int d\mathbf{r}\, e^{i\mathbf{k}\mathbf{r}} V(r), \qquad (9)$$

whence $m = 2(\alpha - d)$. In particular, in the case of Coulomb interaction ($\alpha = 1$, $m = -4$), distributions with $s_1 = -5/4$ and $s_0 = -3/4$ are obtained. The distribution with constant energy flux ($s = s_1$) is then local, i.e., the collision integral converges at both small and large momenta. In "Kolmogorov" states of this kind, the Landau damping and the particle "runaway" increase, the Lawson criterion becomes lower, etc., i.e., those properties which are sensitive to the presence of the "tail" particles change in comparison with the equilibrium values. In the case of a plasma, the dynamic polarization of the medium assumes an important role. The Lennard-Balescu kinetic equation that takes this into account[6,7] takes the form (1), but the effective matrix element, in contrast to (9), now depends both on the momentum transfer and on the energy transfer $\hbar\omega = E_1 - E_3$. In a nonrelativistic plasma we have for the interaction of particles a and b

$$V(\omega, \mathbf{k}) = e_a e_b / k_i k_j \epsilon_{ij}(\omega, \mathbf{k}), \qquad (10)$$

where ϵ_{ij} is the dielectric tensor of the medium with allowance for dispersion.[8] In the region of static screening we have here $\epsilon \sim (kr_D)^{-2}$ and $m = 0$. When exchange of virtual low-frequency plasmons predominates we have $\epsilon \sim -(\omega_p/\omega)^2$ and $m = 4$ (in the relativistic limit, m is a multiple of 4). The distributions become softer in comparison with the Coulomb distributions. In a relativistic plasma, the matrix element is expressed in terms of the Fourier component of the Green's tensor $\hat{G}$ of Maxwell's equations, and the Klimontovich-Silin kinetic equation[6,7] retains the form (1), where

$$V(\omega, \mathbf{k}) = 4\pi e_a e_b v_i^a v_j^b \hat{G}_{ij}^{ab}(\omega, \mathbf{k})/c^2 \qquad (11)$$

($\mathbf{v}$ is the particle velocity), with $\hat{G}$ expressed in the isotropic case in terms of the longitudinal and transverse dielectric constants ϵ^l and ϵ^{tr}, respectively, while at $\epsilon^l = \epsilon^{tr} = 1$ we get the Belyaev-Budker equation. In an ultrarelativistic plasma, this interaction corresponds to a homogeneity exponent $m = -4$ (Coulomb), $m = 0$ (Debye screening), etc., with m a multiple of 2. The homogeneity exponent can change also in a magnetic field [if it is not large enough to violate (1) or (5)]. We present the exponents for the differentiated particle flux $I(E) = vn_p(E) \sim E^{-\gamma}$ ($g(E)$ is the density of states). In the nonrelativistic case ($\beta = 2$) we have $\gamma^{nonrel} = -(1 + S)$, and in the ultrarelativistic case ($\beta = 1$) we have $\gamma^{rel} = -(2 + S)$. At $d = 3$ we get $\gamma_1^{rel} = m/2 + 5/2$ and $\gamma_0^{rel} = m/2 + 2$. Attention is called to the proximity of the calculated spectra to those observed for cosmic rays.

We now find the small deviations from the single-flux power-law solutions (8), which cause I_{col} to vanish[4]

$$n_0 = E^{s_0}[1 + E^{-1}\delta\mu_0 + \frac{\mathbf{p}}{p^2}\delta\mathbf{u}], \quad n_1 = E^{s_1}[1 + E\delta\mu_1 + E(\mathbf{p}\delta\mathbf{u}), \qquad (12)$$

and also the solution with small fluxes in the "plateau" region (cf.[4]):

$$n_p = 1 + p^{-(m+3d)}\left[E\delta\mu_0 + \delta\mu_1 + E\frac{\mathbf{p}\delta\mathbf{u}}{p^2}\right]. \qquad (13)$$

Here $\delta\mu_0$ is proportional to the energy flux, $\delta\mu_1$ to the particle flux, and $\delta\mathbf{u}$ to the momentum flux.

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