Effect of viscosity on the character of the cosmological singularity

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We consider the character of the solutions for a homogeneous anisotropic cosmological model of Bianchi type I, with allowance for the second viscosity. It is shown that the viscosity is unable to eliminate the cosmological singularity, in contrast to the previously investigated Friedmann case. The model considered in the present paper has the remarkable property that near the cosmological singularity the gravitational field produces matter within the framework of the classical (rather than quantum) theory.

In a recent paper, Murphy[1] cites an example of an exactly solvable cosmological Friedmann model with allowance for second viscosity, the action of which eliminates the cosmological singularity. We shall show that this effect is unstable and vanishes on going to anisotropic models. In the anisotropic case, the viscosity is unable to eliminate the singularity.

Just as in[1], we confine ourselves to inclusion of only the second viscosity, the coefficient of which is proportional to the energy density (it can be shown that the first viscosity does not alter the results). We consider a homogeneous cosmological model of the type I:

$$-ds^2 = -dt^2 + R_1^2(t)\,dx^2 + R_2^2(t)\,dy^2 + R_3^2(t)\,dz^2. \qquad (1)$$

When second viscosity is taken into account, the energy-momentum tensor is $T_{ik} = (\epsilon + p')u_i u_k + p' g_{ik}$, where $p' = p - \zeta u^k_{;k}$. The pressure p and the viscosity coefficient ζ are assumed to be proportional to the energy density ϵ:

$$p = (\gamma - 1)\epsilon,\ \zeta = a\epsilon,\ (1 \leqslant \gamma \leqslant 2,\ a \geqslant 0,\ a,\ \gamma = \text{const}). \qquad (2)$$

In the co-moving system we have $u^0 = 1$, $u^\alpha = 0 (u^0 u_\infty = -1)$, and the components of T_i^k are $T_0^0 = -\epsilon$, $T_\alpha^0 = 0$, and $T_\alpha^\beta = p'\delta_\alpha^\beta$. It will be convenient to use the notation

$$R_1 R_2 R_3 = R^3, \quad (\ln R)^{\cdot} = H \qquad (3)$$

We than obtain $p' = \epsilon(\gamma - 1 - 3\alpha H)$. Writing down now the system of Einstein's equations, we note that the $\alpha\beta$ components of these equations admit of the first two integrals, and as a result the total system of the equations of gravitation can be written in the form (4—5)

$$(\ln R_1)^{\cdot} = H + s_1 R^{-3}, \quad (\ln R_2)^{\cdot} = H + s_2 R^{-3}, \quad (\ln R_3)^{\cdot} = H + s_3 R^{-3} \qquad (4)$$

where the constants s_α are connected by the conditions $s_1 + s_2 + s_3 = 0$. The functions $R(t)$ and $\epsilon(t)$ must satisfy the equations

$$\epsilon = 3H^2 - \frac{4s^2}{3} R^{-6}, \quad \dot{\epsilon} = 3\epsilon H(3aH - \gamma), \qquad (5)$$

where $s^2 = 3(s_1^2 + s_2^2 + s_3^2)/8$. The character of the solutions of the system (5) can be easily explained by the qualitative theory of dynamic systems, using the phase plane (H, ϵ). It follows from (5) that

$$\dot{H} = \epsilon\left(\frac{2-\gamma}{2} + \frac{3a}{2}H\right) - 3H^2, \qquad \dot{\epsilon} = 3\epsilon H(3aH - \gamma). \qquad (6)$$

The system (6) has a singular saddle point at $H = \gamma/3\alpha$, $\epsilon = \gamma^2/3\alpha^2$ (the characteristic numbers are $\lambda_1 = \gamma^2/2\alpha$ and $\lambda_2 = -2\gamma/\alpha$). It is easy to show that the exact equa-

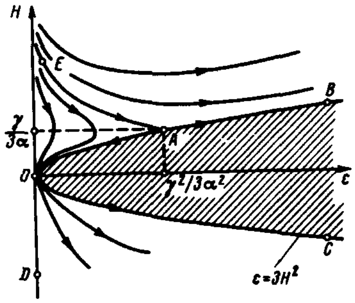

FIG. 1.

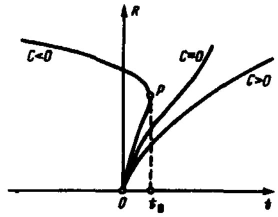

FIG. 2.

tion of the first separatrix is $\epsilon = 3H^2$. The exact equation of the second separatrix can be indicated only in the particular case $\gamma = 2$, namely, this equation is given in parametric form by formula (9) at $C = 0$, but in the general case, when $\gamma \neq 2$, the behavior of the second separatrix remains qualitatively the same. Figure 1 shows the behavior of the integral curves of the system (6). It shows only the curves in the physical region $\epsilon \leq 3H^2$ (see (5)). The parabola $\epsilon = 3H^2$ and the curve EA are the separatrices of the saddle A, and break up the entire aggregate of the solutions into three classes:

1. The solutions in the region EAO describe the evolution of the universe from the cosmological singularity ($H \to +\infty$, $\epsilon \to 0$) to the state of infinite expansion ($H \to 0$, $\epsilon \to 0$). It is easy to indicate the asymptotic forms of the final states. As the point (0,0) is approached, the value of H tends to zero like $1/t$ ($t \to \infty$), and the viscosity terms in the equations are negligibly small. Then $R \sim t^{2/3\gamma}$ and as $t \to +\infty$ we have asymptotically the Friedmann solution.

As $H \to \infty$ (corresponding to $t \to 0$), the viscous term in the equations becomes decisive, and we have here the asymptotic form:

$$R = (2st)^{1/3}, \quad \epsilon = \text{const} \cdot t^{-\gamma} e^{-\alpha/t} \quad (t \to 0) \tag{7}$$

$$(R_1, R_2, R_3) \sim (t^{p_1}, t^{p_2}, t^{p_3}), \tag{8}$$

where the exponents $p_\alpha = 1/3 + s_\alpha/2s$ satisfy the Kasner conditions $\Sigma p_\alpha = \Sigma p_\alpha^2 = 1$.[2] Thus, at $t = 0$ this aggregate of solutions is bounded by the Kasner singularity and, in contrast to the solution of[1], it is impossible to go over to minus infinity in time (with elimination of the singularity). This fact is understandable from the picture of the integral curves (Fig. 1). Murphy's solution[1] is the separatrix OA ($s = 0$), which enters in the unstable saddle point, and the perturbation that turns on the anisotropy ($s \neq 0$) moves the trajectory away to an infinite distance.

2. The regions EAB and DOC contain solutions in which the function $R(t)$ is double-valued in a certain interval. We illustrate all the foregoing with one solution, which can be obtained in quadratures at $\gamma = 2$:

$$H = 2s/3R^3 \,\text{th}\,(asR^{-3} + C), \quad \epsilon = 4s^2/3R^6 \,\text{sh}^2(asR^{-3} + C), \quad (R^3)' = 2s/\text{tn}\,(asR^{-3} + C). \tag{9}$$

Depending on the constant C, we obtain the three types of solutions shown in Fig. 2. At $C > 0$ the solutions correspond to trajectories that lie in the region EAO. At $C = 0$ we obtain the separatrix AE. The double-valued solutions from the origin to the point P are represented by the trajectories in the region EAB, and from the point P to infinity they are represented by the trajectories from DOC. The point P corresponds to the singularity $H \sim \pm (t_0 - t)^{-1/2}$, $\epsilon \sim (t_0 - t)^{-1}$.

It is easy to show that the cosmological singularity at $t = 0$ remains also in the Bianchi IX model. The energy density ϵ then tends rapidly to zero following a law similar to (7), and an oscillatory regime is produced in place of the Kasner singularity.[2]

The vanishing of the energy density at the singular point at $t = 0$ is a qualitatively new element in the behavior of the cosmological solution. As $t \to 0$, the energy density is $\epsilon \sim e^{-\alpha/t}$ and is exponentially small, and then during the expansion process the energy density increases, reaches a maximum, after which it decreases asymptotically as $t \to \infty$ in accord with the Friedmann law. Thus, the cosmological model with viscosity, which is considered in this paper, has the following remarkable property: near the singularity, the gravitational field produces matter within the framework of classical (and not quantum[3,4]) theory. It must be stipulated, of course, that this conclusion was obtained for a special model, in which the second-viscosity coefficient was assumed to be proportional to the energy density. In addition, it must be borne in mind that dissipation can be taken into account with the aid of one viscosity coefficient only when the terms with the higher-order derivatives of the velocity are small. One can hope, however, that the kinetic coefficients of higher order will be proportional to the energy raised to a power larger than the first. In this case, these terms would turn out to be negligibly small near the singularity, in view of the exponential smallness of ϵ as $t \to 0$. In one way or another, the considered model possibly gives an idea of what might have happened during the creation and evolution of our universe.

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