Gravitational instability of the universe filled with a scalar field

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A self-consistent problem involving the behavior of small perturbations in an isotropic homogeneous universe filled with a scalar field is considered. Solutions describing the evolution of perturbations in the case of an arbitrary scalar-field potential are obtained.

The inflationary scenarios of the evolution of the universe[1–3] have made it possible to obtain an elementary-perturbation spectrum for the formation of galaxies from initial quantum fluctuations.[4–11] In the scalar-field models with a Coleman-Weinberg potential, the perturbations at the de Sitter stage were studied in Refs. 7–11. In these studies, however, the metric perturbations occurring after the decay of the de Sitter stage were estimated by qualitative methods.

In this letter we solve the self-consistent problem (with allowance for the metric fluctuation) involving the behavior of the perturbations at all stages of the evolution of a homogeneous isotropic universe filled with a scalar field with an arbitrary potential $V(\varphi)$.

We assume that the total action of the scalar and gravitational fields is given by

$$S = \frac{1}{16\pi G} \int R\sqrt{-g}\, d^4x + \int \left[\frac{1}{2}\varphi_{;i}\varphi^{;i} - V(\varphi)\right]\sqrt{-g}\, d^4x . \tag{1}$$

Here the speed of light is $c = 1$. We can write the metrics of a plane, homogeneous, isotropic universe with small, scalar perturbations as[13]

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)a^2(t)\delta_{\alpha\beta}\, dx^\alpha dx^\beta , \tag{2}$$

since in the first order in the field perturbations $\delta\varphi$

$$\varphi(\mathbf{x}, t) = \varphi_0(t) + \delta\varphi(\mathbf{x}, t) \tag{3}$$

for a scalar field $\delta T^\alpha_\beta = 0$, where $\alpha \neq \beta$. In zeroth-order perturbation theory, Einstein's equations reduce to the following relations for the scale factor $a(t)$:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\varphi_0^2 + V(\varphi_0)\right), \tag{4}$$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 = 8\pi G\left(-\frac{1}{2}\dot\varphi_0^2 + V(\varphi_0)\right), \tag{5}$$

where the dot means differentiation with respect to time t. From (4) and (5) we find

$$\left(\frac{\dot a}{a}\right)^{\cdot} = -4\pi G\dot\varphi_0^2 , \tag{6}$$

and the equation for the homogeneous field mode is

$$\ddot\varphi_0 + 3\frac{\dot a}{a}\dot\varphi_0 + V_{,\varphi}(\varphi_0) = 0, \qquad V_{,\varphi} = \frac{\partial V}{\partial\varphi} . \tag{7}$$

Linearizing Einstein's equations with respect to $\varnothing$ and $\delta\varphi$, we find a system of equations for the perturbations,

$$\frac{1}{a^2}\Delta\varnothing - 3\frac{\dot a}{a}\dot\varnothing - 3\left(\frac{\dot a}{a}\right)^2\varnothing = 4\pi G\,[\dot\varphi_0\delta\dot\varphi - \dot\varphi_0^2\varnothing + V_{,\varphi}(\varphi_0)\delta\varphi\,], \tag{8}$$

$$\frac{1}{a}(a\varnothing)^{\cdot}_{,\beta} = 4\pi G\,(\dot\varphi_0\delta\varphi)_{,\beta} , \tag{9}$$

$$\ddot\varnothing + 4\frac{\dot a}{a}\dot\varnothing + 2\left(\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2\right)\varnothing = 4\pi G\,[\dot\varphi_0\delta\dot\varphi - \dot\varphi_0^2\varnothing - V_{,\varphi}(\varphi_0)\delta\varphi]. \tag{10}$$

Subtracting Eq. (8) from (10) and using (7) and (9), we find that the potential $\varnothing$ satisfies the equation

$$\ddot\varnothing + \left(\frac{\dot a}{a} - 2\frac{\ddot\varphi_0}{\dot\varphi_0}\right)\dot\varnothing - \frac{1}{a^2}\Delta\varnothing + 2\left(\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 - \frac{\dot a}{a}\frac{\ddot\varphi_0}{\dot\varphi_0}\right)\varnothing = 0. \tag{11}$$

Introducing the conformal time $\eta = \int dt/a$ instead of t and using (6), we find the following expression for $u = a/\varphi'\varnothing$:

$$u'' - \Delta u - \frac{(a'/a^2\varphi'_0)''}{(a'/a^2\varphi'_0)}\,u = 0, \tag{12}$$

where the prime means differentiation with respect to η. Equation (12) can easily be solved in the asymptotic limit. Let us consider a plane wave with a wave vector k, i.e., $\varnothing$, $\delta\varphi$, $u \propto e^{i\mathbf{kx}}$. Solving Eq. (12) and expressing the result in terms of t, after straightforward calculations, we finally find

$$\varnothing = A\left(1 - \frac{\dot a}{a^2}\int^t a\,dt\right), \tag{13}$$

$$\delta\varphi = A\,\dot\varphi_0\frac{1}{a}\int a dt \tag{14}$$

for the long-wavelength perturbations for which $k^2 \ll [(a'/a^2\varphi'_0)''/(a'/a^2\varphi'_0)]$. The integration constant in (13) and (14) corresponds to the incident mode. In the case of perturbations with scale dimensions much smaller than the horizon, for which $k^2 \gg [(a'/a^2\varphi'_0)''/a'/a^2\varphi'_0]$, we find

$$\varnothing = \dot\varphi_0\left(C_1\sin\left(k\int\frac{dt}{a}\right) + C_2\cos\left(k\int\frac{dt}{a}\right)\right)e^{i\mathbf{kx}} , \tag{15}$$

$$\delta\varphi \simeq \frac{1}{4\pi G}\,\frac{k}{a}\left(C_1\cos\left(k\int\frac{dt}{a}\right) - C_2\sin\left(k\int\frac{dt}{a}\right)\right)e^{i\mathbf{kx}}. \tag{16}$$

Let us now focus attention on the study of the long-wavelength perturbations. From (13) and (14) we can derive for these perturbations a relationship between $\varnothing$ and $\delta\varphi$:

$$\varnothing = \left[\frac{a}{\int^t a\,dt}\,\frac{\delta\varphi}{\dot{\varphi}_0}\right]\left(1 - \frac{\dot{a}}{a^2}\int a\,dt\right). \tag{17}$$

In the case of long-wavelength perturbations, the quantity

$$A = \frac{a}{\int^t a\,dt}\,\frac{\delta\varphi}{\dot{\varphi}_0}$$

is constant at any stage of evolution of the universe at which the approximation for the perturbations considered by us is appreciable.

Let us now examine a specific example. We assume that the potential $V(\varphi)$ is such that is has a minimum near the point φ_m. A homogeneous scalar field oscillates rapidly near the minimum and the universe expands (to within small oscillations of the scale factor) in the manner of a universe filled with dust; i.e., $a(t)\propto t^{2/3}$ (Refs. 15 and 16). Using (14), we then find

$$\varnothing \simeq \frac{3}{5}A + \frac{B}{t^{5/3}}; \tag{18}$$

i.e., the behavior of long-wavelength perturbations of the metric in this case is similar to the behavior of the perturbations in a dust-filled universe.[13,14] If the scalar field subsequently breaks up into ultrarelativistic particles, the metric perturbations remain the same. The constant A, as was already noted, can be expressed in terms of the scalar-field perturbations $\delta\varphi$,

$$A = \frac{a}{\int^t a\,dt}\,\frac{\delta\varphi}{\dot{\varphi}_0}. \tag{19}$$

Let us assume that the "powdered" stage was preceded by the "quasi-de Sitter" stage, at which $a\propto e^{Ht}$, where the Hubble constant H remains essentially constant throughout this stage (this situation is encountered in the case of a Coleman-Weinberg potential[2]). The amplitude of the undamped mode of the metric perturbations, $\varnothing$, and the "densities" $\delta T_0^0/T_0^0$ at the powdered stage and the subsequent ultrarelativistic stage can then be expressed in terms of scalar-field perturbations at the quasi-de Sitter stage as

$$\varnothing = \frac{3}{5}H\left(\frac{\delta\varphi}{\dot{\varphi}_0}\right)_{\text{d.S.}} \qquad \frac{\delta T_0^0}{T_0^0} = -2\varnothing = -\frac{5}{6}H\left(\frac{\delta\varphi}{\dot{\varphi}_0}\right)_{\text{d.S.}}, \tag{20}$$

where $(\delta\varphi/\dot{\varphi}_0)_{\text{d.S}}$ is estimated at an arbitrary moment of time at the de Sitter stage when the perturbation considered by us is larger than the horizon. This result coincides within a numerical factor with the estimates obtained in Refs. 7–11. The amplitude $\delta\varphi$ at the de Sitter stage under consideration, which corresponds to the initial quantum fluctuations, can be determined by quantizing the scalar field without regard for the inverse effect of the metric perturbations, since these perturbations in it are negligible, as we can see by analyzing (13) and (15). In random-inflation scenarios,[12] Eq. (19) should generally be used to estimate the perturbations, since the Hubble constant changes markedly during an inflation period throughout the inflationary stage.

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[1]A. Guth, Phys. Rev. D **23**, 347 (1981).
[2]A. D. Linde, Phys. Lett. **108B**, 389 (1982).
[3]A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[4]G. V. Chibisov and V. F. Mukhanov, "Galaxy formation and phonons," Preprint No. 162, P. N. Lebedev Physics Institute.
[5]V. F. Mukhanov and G. V. Chibisov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 549 (1981) [JETP Lett. **33**, 532 (1981)].
[6]V. F. Mukhanov and G. V. Chibisov, Zh. Eksp. Teor. Fiz. **83**, 475 (1982) [Sov. Phys. JETP **56**, 258 (1982)].
[7]A. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
[8]S. W. Hawking, Phys. Lett. **115B**, 295 (1982).
[9]J. P. Bardeen, P. Steinhardt, and M. Turner, Phys. Rev. D **26**, 679 (1983).
[10]A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[11]R. Brandenberger and R. Kahn, "Cosmological perturbations in inflationary universe," 1983, Harvard University preprint.
[12]A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[13]G. V. Chibisov and V. F. Mukhanov, "Theory of Relativistic Potential: Cosmological Perturbations," 1983, Preprint No. 154, P. N. Lebedev Institute.
[14]E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **16**, 587 (1946).
[15]Ya. B. Zel'dovich, Soviet Sci. Rev., Ser. E, Astrophys., ed. R. A. Sunyaev, Vol. 6, 1985.
[16]M. Yu. Khlopov, and B. A. Malamed, and Ya. B. Zel'Dovich, M.N.R.A.S. **214**, 100 (1985).

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