Effect of quantum fluctuations on the shape of an instanton

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Quantum effects strongly influence the shape of an instanton. In particular, for an instanton with a dimension small in comparison with the confinement radius the power-law asymptotic behavior of the classical solution at large distances from the center of the instanton is replaced by a Gaussian decay of the profile function.

The conventional approach in calculating the instanton contributions to various vacuum expectation values is to carry out an expansion in small deviations from the mean field, which satisfies the classical Yang-Mills equations.[1–3] This expansion clearly "does not work" for instantons with dimensions comparable to the confinement radius R_c, in which case nonperturbative effects become important and change both the shape of the instantons and their size distribution. Surprisingly, however, even for a small instanton quantum effects cause a substantial change in the profile function at large distances from the center of the instanton ($R_c^2 \gg x^2 \gtrsim \rho^2/\alpha_s$, where ρ is the scale dimension of the instanton, and α_s is the gluon-dynamic coupling constant).

The simplest way to incorporate the effect of quantum fluctuations on the shape of an instanton is to use the method of an effective action. An effective action can be used to derive an equation for the mean field. In the case of an instanton, however, there are some specific features that must be dealt with: The existence of zero-point modes in the field of the instanton makes it necessary to use "transverse" Green's functions in the loop expansion for the effective action. The corresponding collective coordinates [x_0, the "center of mass" of the instanton; ρ, its scale dimension; and $S(x)$, the gauge phase] remain quantum degrees of freedom of the instanton field. Accordingly, in contrast with the standard approach of using an effective action, in this case we should find the contribution of an instanton with given values of x_0, ρ, and $S(x)$, and the final results are then found by taking an average over these collective coordinates. These coordinates are ordinarily introduced through the use of the solution of the classical equations:

$$A^a_\mu(x) = 2\bar{\eta}^a_{\mu\nu} \frac{x_\nu}{x^2} f_0\left(\frac{x^2}{\rho^2}\right), \quad f_0(\xi) = \frac{1}{1+\xi^2}, \quad \xi_\mu = \frac{x_\mu}{\rho}. \tag{1}$$

The incorporation of quantum effects results in a change in the profile function $f(\xi^2)$ of the mean gluon field. We should thus single out the collective coordinates, using specifically this future mean field, rather than its classical part. It is not difficult to see that in order to derive equations of motion from the effective action, we need to impose restrictions on possible variations of the mean field; specifically, the variations $\delta A_\mu(\xi)$ must not change x_0, ρ, or $S(x)$. In other words, the orthogonality conditions

$$\int d^4\xi \, \mathrm{tr}\,[\delta A_\mu \overset{A}{D_l} \bar{A}_\mu] = 0, \quad A_\mu = A^a_\mu \frac{\tau^a}{2} \tag{2}$$

must be satisfied, where $\hat{D}_i\bar{A}_\mu(\xi)$ are the changes in the mean field $\bar{A}_\mu$ due to the change in the collective coordinates. These restrictions lead to a modification of the equations of motion which follow from the requirement that the effective action $S_{\rm eff}(A_\mu)$ be minimized:

$$\frac{\delta S_{eff}}{\delta A_\mu(\xi)}\bigg|_{A_\mu=\bar{A}_\mu} = \sum_i \lambda_i \hat{D}_i \bar{A}_\mu(\xi) \tag{3}$$

where λ_i are the corresponding Lagrange multipliers. To determine them, we multiply Eq. (3) by $\hat{D}_j\bar{A}_\mu(\xi)$ and integrate over ξ. By virtue of the orthogonality of the various $\hat{D}_i\bar{A}_\mu(\xi)$, we find

$$\lambda_i = \frac{1}{N_i^2}\int d^4\xi \, {\rm tr}\left\{\frac{\delta S_{eff}}{\delta A_\mu(\xi)}\bigg|_{A_\mu=\bar{A}_\mu} \hat{D}_i\bar{A}_\mu(\xi)\right\}, \tag{4}$$

$$N_i^2 = \int d^4\xi \, {\rm tr}\,\{\hat{D}_i\bar{A}_\mu(\xi)\}^2 . \tag{5}$$

From (4) we see that the λ_i are related to the change in $S_{\rm eff}$ upon the corresponding symmetry transformation. Accordingly, if the symmetry is not broken by the quantum anomalies, we find that the corresponding values of λ_i are zero. In our case, the only broken symmetry is the gauge symmetry, and the only nonvanishing Lagrange multiplier λ_ρ is determined by

$$\lambda_\rho = \frac{1}{N_\rho^2}\int d^4\xi \, {\rm tr}\left\{\frac{\delta S_{eff}}{\delta A_\mu(\xi)}\bigg|_{A_\mu=\bar{A}_\mu} \hat{D}_\rho\bar{A}_\mu(\xi)\right\} = \frac{1}{N_\rho^2}\int d^4\xi \, \theta_{\mu\mu}(\xi), \tag{6}$$

where $\hat{D}_\rho\bar{A}_\mu = (1+\xi_\nu\partial_\nu)\bar{A}_\mu(\xi)$, and $\theta_{\mu\mu}(\xi)$ is the trace of the energy-momentum tensor. Using the standard expression for a conformal anomaly,[4] we find

$$\lambda_\rho = \frac{1}{N_\rho^2}\int d^4\xi \, \frac{\beta(\alpha_s)}{\alpha_s^2}\, {\rm tr}\, F_{\mu\nu}^2 , \tag{7}$$

where α_s depends on the scale dimension l and on the field $F^2 = 2\,{\rm tr}\,F_{\mu\nu}^2$.

To make Eq. (3) meaningful, we must calculate $S_{\rm eff}$ or develop a model for it:

$$S_{eff} = \frac{b}{8\pi}\int d^4\xi \, \frac{1}{\alpha_s}\, {\rm tr}\, F_{\mu\nu}^2 .$$

For a small instanton the field near the center of the instanton is strong, and in addition it can be assumed quite accurately to be uniform, since the uniformity condition[5] $F'_\xi F^{-3/2} \ll 1$ holds by virtue of the large numerical factor $F^2(0) = 192$. In this case we can use the familiar expression[6] in the leading-log approximation:

$$\frac{1}{\alpha_s} = \frac{b}{4\pi}\ln\frac{F}{\Lambda^2}, \quad b = \frac{11}{3}N_c, \quad \Lambda \sim 100 \ \text{MeV}.$$

Far from the center of the instanton ($R^2_{c/\rho^2} \gg \xi^2 \gg 1$) the field becomes weak, and we can use the customary expression

$$\frac{1}{\alpha_s} = \frac{b}{4\pi} \ln \frac{1}{\xi^2 \rho^2 \Lambda^2} .$$

The following interpolation formula, which joins the two asymptotic expressions, is quite accurate:

$$\frac{1}{\alpha_s} \approx \frac{b}{4\pi} \ln \frac{F + \frac{1}{\xi^2}}{\rho^2 \Lambda^2} . \tag{8}$$

Substituting (8) into (3), and using the single-loop expression for $\lambda_\rho = -(b/16\pi^2)$ in the instanton ansatz,

$$\bar{A}^a_\mu = 2\bar{\eta}^a_{\mu\nu} \frac{\xi_\nu}{\xi^2} f(\xi^2),$$

we find an equation for the profile function f:

$$\ddot{f} - (2f^3 - 3f^2 + f) - \frac{\dot{\alpha}_s}{\alpha_s} \dot{f} = - \frac{b\alpha_s}{8\pi} e^t \dot{f}, \tag{9}$$

$$t = \ln \xi^2, \quad \frac{1}{\alpha_s} = \frac{b}{4\pi} \left\{ \ln \frac{1}{\rho^2 \Lambda^2} \{ [96 (\dot{f}^2 + f^2 (1-f)^2]^{1/2} \} - t \right\} .$$

The boundary conditions $f(0) = 1, f(\infty) = 0$ are dictated by the requirement that the topological charge be unity:

$$Q[\bar{A}_\mu] = \frac{1}{16\pi^2} \int d^4\xi \, \mathrm{tr}\, F_{\mu\nu} \widetilde{F}_{\mu\nu} = 1.$$

We see from (9) that under the condition $\xi^2 \gg 1$ we have the asymptotic behavior

$$f((\xi^2) \sim \frac{1}{\xi^2} \exp\left\{ - \frac{b}{8\pi} \alpha_s (\xi \rho \Lambda) \xi^2 \right\};$$

at $\xi^2 \ll 1$, on the other hand, we can use the classical solution, (1), and the interpolation formula

$$f(\xi^2) = \frac{1}{1 + \xi^2} \exp\left\{ - \frac{b}{8\pi} \alpha_s (\xi \rho \Lambda) \xi^2 \right\},$$

which gives an approximate description of the behavior of the profile function at $\xi^2 \ll R^2_{c/\rho^2}$. We thus see that the quantum effects cause a significant decrease in $f(\xi^2)$ at distances $R^2_{c/\rho^2} \gtrsim \xi^2 \gtrsim 1/\alpha_s$.

The effects discussed here become particularly important for instantons with scale dimensions $\rho \sim R_c$. It may be that the incorporation of these effects will lead to a solution of the problem of calculating the instanton contributions to various physical quantities.

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