Phase transition with an infinite number of order parameters in the $U(\infty)$ lattice gauge theory

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(Submitted 1 April 1985)
Pis'ma Zh. Eksp. Teor. Fiz. **41**, No. 10, 442–445 (25 May 1985)

The $U(N)$ lattice gauge theory identifies N gauge-invariant parameters in the limit $N\to\infty$. These parameters describe the phase transition from the strong-coupling region to the weak-coupling region. A strong-coupling-expansion method is used to calculate the effective potential of the theory in terms of these parameters. A single-placquet model and a second-order expansion for a D-dimensional gauge theory are used to test the effectiveness of the method.

It is more difficult, as we know, to use the strong-coupling expansion in $U(N)$, $N>3$ lattice gauge theories because of a first-order phase transition which renders the weak-coupling phase inaccessible.[1]

In this letter (see also Ref. 2) we describe a method which can be used to modify the strong-coupling expansion and to uniquely describe both phases.

We will describe this method by using as an example the single-placquet model,[1] whose partition function is

$$Z = \int d(U) \exp\left[N\beta \,\mathrm{tr}\,(U^+ + U)\right], \tag{1}$$

where $d(U) - U(N)$ is the Haar's measure. We will parametrize the unitary condition $U^+U=1$ by means of a Lagrangian multiplier—a skew-Hermitian matrix α (Ref. 2):

$$Z = \int d\alpha \int d^2U \exp\left\{ N\,\mathrm{tr}\left[\alpha - \alpha U^+ U + \beta(U^+ + U)\right]\right\}. \tag{2}$$

Integrating over U and U^+ and using the "polar" representation for α

$$\alpha_{ij} = \sum_k \Omega^+_{ik} \lambda_k \Omega_{kj}, \tag{3}$$

where Ω is a unitary $U(N)$ matrix, we find

$$Z = \int \prod_{i=1}^{N} d\lambda_i \exp\left[N \sum_{i=1}^{N} (\lambda_i - \ln\lambda_i + \beta^2 \lambda_i^{-1}) + \sum_{i\neq j} \ln|\lambda_i - \lambda_j|\right]. \tag{4}$$

Here the expression within the exponential function in (4) is understood as the effective potential of $N\,\lambda_i$ variables, which is a first-order expansion in β^2 (that yields an exact result in this case). By introducing the normalized eigenvalue density $\rho(\lambda)$ we can now use the standard methods[3] to accurately analyze the problem in the limit $N \to \infty$ and to find for this potential an equation for the stationary point of λ_i^* in (4):

$$2 \int \frac{\rho(\lambda')d\lambda'}{\lambda - \lambda'} = -1 + \frac{1}{\lambda} + \frac{\beta^2}{\lambda^2}. \tag{5}$$

Solving (5) by the methods of Ref. 3, we find for the placquet energy, for example,

$$E_{pl}(\beta) = \langle \frac{\mathrm{tr}}{2N}(U^+ + U) \rangle_U = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^{*-1}, \tag{6}$$

a solution with two branches—a strong and a weak coupling:

$$E_{pl} = \begin{cases} \beta, & \beta \leqslant \frac{1}{2} \\ 1 - \frac{1}{4\beta}, & \beta \geqslant \frac{1}{2}, \end{cases} \tag{7}$$

which are divided by the Gross-Witten phase transition.[1] We thus see that the effective potential describes in an effective way both phases of the model in terms of λ_i in (4).

Our main task is to calculate the effective potentials as functions of λ_i by expanding the potential in powers of β. In contrast with the standard strong-coupling expansion, we can describe both phases by singling out the free parameters λ_i, which play the role of N order parameters in the D-dimensional gauge theory.

To verify this program, we will examine the same single-placquet model in the form

$$Z = \int d(U) \int d(V) \exp\left[\beta N \mathrm{tr}(UV^+ + VU^+)\right]. \tag{8}$$

Since our aim here is to give an example, we will not make use of the obvious choice of the gauge $V = I$.

Introducing the Lagrangian multipliers α_1 and α_2 for the U and V matrices and carrying out a Gaussian integration over U and V, we find

$$Z = \int d\alpha_1 \int d\alpha_2\, e^{N\,\mathrm{tr}(\alpha_1 + \alpha_2) - \mathrm{tr}_1 \mathrm{tr}_2 \ln(\alpha_1 \otimes \alpha_2 - \beta^2)}. \tag{9}$$

Transforming to the "polar" variables according to Eq. (3) for α_1 and α_2 and expanding the logarithm in (9) in powers of β^2, we find

$$Z = \int \prod_{k=1}^{2} \prod_{i=1}^{N} d\lambda_i^{(k)} \exp\left\{ \sum_{k=1}^{2} \left[\left(\sum_i \lambda_i^{(k)} - \ln \lambda_i^{(k)} \right) + \sum_{i \neq j} \ln |\lambda_i^{(k)} - \lambda_j^{(k)}| \right] + \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n} \prod_{k=1}^{2} \left(\sum_i (\lambda_i^{(k)})^{-n} \right) \right\}. \tag{10}$$

In the limit $N \to \infty$, we find again the equation which describes the stationary configuration of λ_i^* (from the symmetry $\lambda_i^{(1)*} = \lambda_i^{(2)*}$)

$$2\int \frac{d\lambda' \rho(\lambda')}{\lambda - \lambda'} = -1 + \sum_{n=0}^{M} \beta^{2n} \mu_n \lambda^{-n-1} . \tag{11}$$

In this equation we use the notation

$$\mu_0 = 1$$

$$\mu_n = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^{-n} = \int d\lambda \rho(\lambda) \frac{1}{\lambda^n} . \tag{12}$$

In an exact equation we should set $M = \infty$ in (11); however, in our analysis of Eq. (11) M is finite, and the right side of this equation is an approximate calculation based on the expansion in a series in β^2.

We find that even if M is finite, Eq. (11) describes two phases: a high-temperature phase in which

$$\mu_1 = 1, \tag{13}$$

$$\mu_n = 0, \quad n = 2, 3, \ldots \tag{14}$$

and a low-temperature phase, which we have analyzed numerically using the procedure of Ref. 3 for finding self-consistent equations with μ_n. The results of calculation of the quantity

$$E_{pl}(\beta) = \frac{1}{2N} \frac{\partial \ln Z}{\partial \beta} \approx \sum_{n=1}^{M} \beta^{2n-1} \mu_n^2 \tag{15}$$

are shown in Fig. 1 for various values of M. Expression (15) is a good approximation of the exact solution of (7) to within a certain value of β, at which the agreement worsens markedly. The agreement was found to improve with increasing value of M.

A similar situation is encountered in a real D-dimensional gauge theory, although the transition from one phase to another is a first-order transition, consistent with the known Monte Carlo data.[4] At fixed values of λ_i (which are identical at all points in the lattice if we work from the spatial symmetries of the steady-state solution) we can write the placquet energy within an error margin of order $O(\beta^4)$

$$E_{pl}(\lambda, \beta) = \beta \mu_1^4 + \beta^3 [4(2D-3) \mu_1^6 \mu_2 + 6 \mu_1^4 \mu_2^2] . \tag{16}$$

Note that λ_i is a gauge-invariant variable. The relevant steady-state equation is

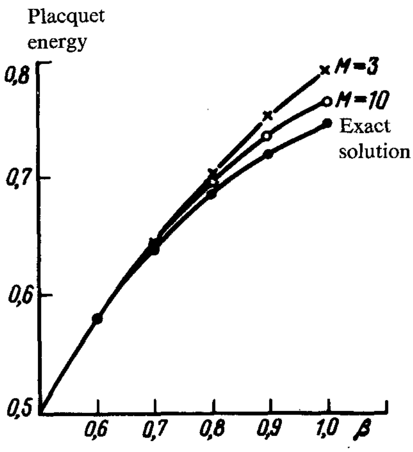

FIG. 1. Results for a single-placquet model with two matrices in the weak-coupling region. These results were obtained by modifying the expansion with three and ten terms (an exact solution can be reproduced in the strong-coupling region).

$$2 \fint \frac{\rho(\lambda) d\lambda}{\lambda - \lambda'} = -1 + \lambda^{-1} + A\lambda^{-2} + B\lambda^{-3}, \tag{17}$$

where

$$A = (D-1)[2\beta^2 \mu_1^3 + 6(2D-3)\beta^4 \mu_1^5 \mu_2 + 6\beta^4 \mu_1^3 \mu_2^2], \tag{18}$$

$$B = 2(D-1)\beta^4 [(2D-3)\mu_1^6 + 3\mu_1^4 \mu_2]. \tag{19}$$

Using the same procedure to analyze (17)–(19), we find that at $n \geqslant 2$ there is a high-temperature solution $\mu_1 = 1$, $\mu_n = 0$ for all values of β. We easily see, on the other hand, that there is a low-temperature branch at least for large values of β. From (17)–(19) we find the self-consistent solution

$$E_{pl}(\beta) \underset{\beta \to \infty}{\approx} a\beta^{-3/7}, \tag{20}$$

where

$$a = [6(D-1)^{-6}]^{1/7}. \tag{21}$$

A first-order transition between two solutions (17)–(19) typically occurs at β_c, which corresponds to the condition under which the thermodynamic potentials of both phases are equal. Equations (17)–(19) will be analyzed in a comprehensive study.

In summary, we have described a method for analytic calculation of the physical quantities in the $U(\infty)$ lattice gauge theory. In this method the strong-coupling expansion is modified by introducing N gauge-invariant order parameters λ_i which describe both phases of the theory. Further quantitative study of the theory by this method requires the use of a computer.

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Translated by S. J. Amoretty