Wilson loop in gluodynamics

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A correlation function of vacuum fluctuations is introduced, and a specific method is proposed for taking an average over the vacuum state for a Wilson loop in gluodynamics. The expression derived for the Wilson loop has the appropriate physical properties. It can be used to express the parameters of the correlation function in terms of observables.

The Wilson loop $W(C)$ is the simplest gauge-invariant entity which contains the basic information about the vacuum fluctuations in gluodynamics. In this letter we calculate the nonperturbative contribution to

$$W(C) = \frac{1}{N}\langle \mathrm{tr} P \exp\left(i\oint_C dx_\mu A_\mu(x)\right)\rangle \ , \tag{1}$$

where N is the number of colors, $\langle \ldots \rangle$ is a vacuum expectation value and P means an ordering of the gauge field A_μ along a closed contour C in a Euclidean space.

We restrict the analysis to a plane contour C, and we write the expression for $W(C)$ as an integral over the area bounded by contour C:

$$W(C) \approx \frac{1}{N}\langle \mathrm{tr} \exp\left(\frac{i}{2}\int d^2\omega F(\omega)\right)\rangle , \tag{2}$$

where

$$F(\omega) = \lambda^a F^a(\omega), \quad F^a(\omega) = \frac{1}{2} s_{\mu\nu} F^a_{\mu\nu}(\omega),$$

$s_{\mu\nu}$ is a unit area, and the $\lambda^a/2$ are the generators of SU(N). Here we are taking $F(\omega)$ to be a diagonal matrix with eigenvalues $\kappa_i(\omega)$; we can always arrange this situation through a gauge transformation. Expression (2) becomes exact if, in taking the vacuum expectation value, we restrict the class of potentials to those for which it is possible to simultaneously diagonalize the two spatial components lying in the plane of contour C ("diagonalizable configurations"). For sufficiently small contours, for which the functions $F^a(\omega)$ can be assumed constant, expression (2) becomes the expression which Shifman[1] has identified as the sum of the basic contributions to $W(C)$.

In (2) the expectation value is taken not over the ($N^2 - 1$)-st component of F^a, as is customarily done in connection with the use of a factorization hypothesis,[1] but over the ($N - 1$)-st independent eigenvalue κ_i. The values of $\kappa_i(\omega)$, in contrast with $F^a(\omega)$, are gauge-invariant, and $N - 1$ is equal to the number of independent invariants of SU(N). Specifically, we use a Gaussian average and make use of the traceless nature of $F(\omega)$:

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$$\langle\Phi\rangle = \prod_i DK_i(x)\,\delta(\sum_i K_i(x))\exp\left\{-\frac{1}{2}\int d^4x\int d^4y\,K_i(x)C^{-1}(x,y)K_i(y)\right\}\Phi, \qquad (3)$$

which allows us to carry out a path integration. For SU(3) we have

$$C(x,y) = \frac{1}{2}\langle K_i(x)K_i(y)\rangle = -3\langle K_i(x)K_{j\neq i}(y)\rangle = \frac{1}{12}\langle (F^a_{\mu\nu})^2\rangle f((x-y)^2/l^2). \qquad (4)$$

Here we have introduced the correlation function of the vacuum fluctuations, $f(\xi^2)(f(0)=1)$, with a characteristic correlation length l which is defined as the area under the $f(\xi^2)$ curve in terms of the coordinates $|x-y|$. Using (2)–(4), we find the following simple expression for the Wilson loop:

$$W(C) = \exp\left\{-\frac{1}{144}\langle (F^a_{\mu\nu})^2\rangle \int d^2\omega_1\int d^2\omega_2 f((\omega_1-\omega_2)^2/l^2)\right\} \qquad (5)$$

For a rectangular loop with dimensions far larger than the coherence length l, we find the expected shape

$$W(C) = \exp(-\sigma S + aP - b), \qquad (6)$$

where S is the area and P the perimeter. The tension σ is expressed in terms of the vacuum energy density and the correlation length:

$$\sigma = \frac{\pi}{72}\langle (F^a_{\mu\nu})^2\rangle l^2\bar{\xi}, \quad \overline{\xi^n} = \int_0^\infty d\xi\,\xi^n f(\xi^2). \qquad (7)$$

A rough estimate based on the lattice data for σ (Ref. 2) and the value of $\langle (F^a_{\mu\nu})^2\rangle$ from Ref. 3 yields the completely reasonable value $l \approx 0.3$ fm. For the other constants we have

$$a = \sigma l\,\overline{\xi^2}/\pi\bar{\xi}, \quad b = \sigma l^2\,\overline{\xi^3}/\pi\bar{\xi}. \qquad (8)$$

The positive constant a contributes a negative constant increment to the linearly growing potential of the quarkonium, which corresponds to most of the fits in the potential models[4]; it also corresponds to the theoretical considerations of Ref. 5. An accurate independent determination of these constants would make it possible to draw conclusions about the shape of the correlation function $f(\xi^2)$ from its moments $\overline{\xi^n}$.

In the other limiting case, of loops of small dimensions, $\omega \ll l$, we obviously find a universal Gaussian functional dependence on the area:

$$W(C) = \exp\{-\langle (F^a_{\mu\nu})^2\rangle S^2/144\}; \qquad (9)$$

i.e., taking an average over the independent eigenvalues $\kappa_i(x)$ in this case does not lead to the nonphysical negative values of $W(C)$ which arise when an average is taken over the independent components $F^a(x)$ on the basis of a factorization hypothesis.[1] At $N=3$ there is a large quantitative difference between the expectation values of the higher powers of the field for the two averaging methods being compared here:

$$\langle (F_a^2)^m \rangle = \left\langle \left(\frac{1}{2}\kappa_i^2\right)^m \right\rangle = \begin{cases} m!\, \langle F_a^2 \rangle^m & \text{(9a)} \\ \dfrac{(m+3)!}{6\cdot 4^m} \langle F_a^2 \rangle^m & \text{(9b)} \end{cases}.$$

Here (9a) corresponds to the local form of the average in (3), while (9b) corresponds to the analogous Gaussian average over the components F^a. In the limit $N \to \infty$, each of the methods yields $\langle (F_a^2)^m \rangle = \langle F_a^2 \rangle^m$. An important point, however, is that the expansion of $W(C)$ contains the expectation values of only the higher-order powers of each component κ_i^n while the use of a factorization hypothesis actually means ignoring the presence of expectation values of the higher-order powers of the individual components because of the large number of cross terms containing the products of the squares of the various components. As a result, the factorization hypothesis breaks down here.

By expanding function (5) in a series, and then expanding $f(\xi^2)$ in powers of $(\omega_1 - \omega_2)^2$, we can draw conclusions about the derivatives of the correlation function at the origin. For example, restricting the expectation value in (2) to the class of diagonalizable potentials A_μ defined above, we can replace the Laplacian ∂_μ^2 in the expansion of $\langle \kappa_i(\omega_1)\kappa_i(\omega_2) \rangle$ by the covariant quantity D_μ^2, and we can compare the result with the expansion of W in (5). As a result, we find

$$4\langle (F_{\mu\nu}^a)^2 \rangle\, \partial f / \partial(\omega_1 - \omega_2)^2 \approx \langle f^{abc} F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c \rangle. \tag{10}$$

In our case, the contribution of the operators of dimensionality 6 to $W(C)$ is positive if we take $\langle fF^3 \rangle$ from Ref. 3 and allow for the Euclidean nature of the situation. This contribution differs in sign from the result of Ref. 1, and the sign is important for obtaining the expected decrease of the function $f(\xi^2)$ at $\xi^2 = 0$. We are not able at present to determine the absolute value of $\partial f / \partial(\omega_1 - \omega_2)^2$ from (10) because of a discrepancy between the estimates that have been published for the invariant $\langle fF^3 \rangle$.

We note in conclusion that the law of areas embodied in (6) corresponds to quark confinement because of the presence in a vacuum of random and, to a large extent, independent fluxes of the gluon field strength; i.e., it corresponds to the confinement mechanism of Ref. 6.

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