Some representations of an SU(2) expanded supersymmetry group with central charges

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An SU(2) expanded supersymmetry group with central charges Z_1 and Z_2 is analyzed. A study is made of the spin-isospin structure of those irreducible representations for which the momenta P_μ and the charges Z_1 and Z_2 satisfy $P^2 = Z_1^2 + Z_2^2$.

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Fayet[1] and Sohnius[2] have outlined the structure of a hypermultiplet for an SU(2) expanded supersymmetry group with central charges. So far, we do not have a description of the entire series of representations to which this multiplet belongs. In the present letter we analyze the spin-isospin structure of those irreducible representations of this supergroup for which the eigenvalues of the square momentum, P^2, and of the central charges are related by $P^2 = Z_1^2 + Z_2^2$. The Fayet-Sohnius hypermultiplet is the simplest of these representations.

We consider the SU(2) expanded supersymmetry group[3,4] with central charges[5] Z_1 and Z_2. We write the commutation relations for the supersymmetry generators S_α in this case as[1]

$$\{S_\alpha^k, \bar{S}_i^\beta\} = \delta_i^k (\gamma P + Z_1 + i\gamma_5 Z_2)_\alpha{}^\beta, \tag{1.1}$$

$$[M_{\mu\nu}, S_\alpha{}^k] = i\,(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta{}^k, \tag{1.2}$$

$$[I_a, S_\alpha{}^k] = \frac{1}{2} S_\alpha{}^i (\tau_0)_i^k, \tag{1.3}$$

where the S_α^k obey the SU(2)-covariant Majorana condition[4]

$$\bar{S}_k^\alpha \equiv \overline{S_\alpha^k} = \epsilon_{ki}\,(C^{-1}\gamma_5)^{\alpha\beta}\, S_\beta{}^i.$$

In (1.3), I_a are the isospin generators, and $(\tau_a)_i^k$ are the isospin Pauli matrices. We will not reproduce the rest of the well-known relations of this superalgebra.

Let us consider representations for which the eigenvalues of the operators P^2, Z_1, and Z_2 satisfy the relation $P^2 = m^2 = Z_1^2 + Z_2^2$. For such representations, the anticommutator (1.1), expressed in terms of the quantity[7]

$$Q_\alpha = (\exp \frac{i}{2}\gamma_5\,\theta)_\alpha{}^\beta\, (S_\beta{}^1 + i S_\beta{}^2),\; Z_1 + i\gamma_5 Z_2 = Z \exp i\gamma_5\,\theta,$$

takes the following form in the rest frame ($\mathbf{P} = 0$) and in the standard representation for γ matrices:

$$\{Q_\alpha, \bar{Q}^\beta\} = 2m\delta_\alpha^{\ \beta} \qquad \{Q_\alpha, Q_\beta\} = 0 . \tag{2.1}$$

This form corresponds to the algebra of the creation, $\bar{Q}^\alpha = (Q_\alpha)^+$, and annihilation, Q_α, operators for the values $\alpha = 1$ and 2, while the matrix elements of Q_α and $(Q_\alpha)^+$ with $\alpha = 3$ and 4 vanish. We have set $Z = m$ in (2.1). Commutation relations for the nonvanishing generators Q_α ($\alpha = 1,2$) with the generators of the angular momentum $M^n = -\frac{1}{2}\epsilon^{onml}M_{ml}$ and I_a follow from (1.2) and (1.3):

$$[M^n, Q_\alpha] = -\frac{1}{2}(\sigma^n)_\alpha^{\ \beta} Q_\beta ; \quad [I_+, Q_\alpha] = 0,$$
$$[I_-, Q_\alpha] = i\bar{Q}_\alpha ; \qquad [I_2, Q_\alpha] = \frac{1}{2} Q_\alpha , \tag{2.2}$$

where $(\sigma^n)_\alpha^\beta$ are the spin Pauli matrices, $I_\pm = I_3 \pm iI_1$ and $Q^\alpha = \epsilon^{\alpha\beta}Q_\beta$, $\bar{Q}_\alpha = \epsilon_{\alpha\beta}\bar{Q}^\beta$. Relations for $\bar{Q}_\alpha$ are found from (2.2) by taking the Hermitian adjoint. The superspin vectors Y^μ and the superisospin vectors T_a of the small momentum group $P^\mu = (m,0,0,0)$ take the following form for representations with $m = Z$:

$$Y^0 = 0; \quad Y^n = M^n - \frac{1}{4m}\bar{Q}^\alpha(\sigma^n)_\alpha^{\ \beta} Q_\beta ; \quad (n = 1, 2, 3),$$
$$T_- = (T_+)^+ = I_- + \frac{i}{4m}\bar{Q}^\alpha\bar{Q}_\alpha ,$$
$$T_2 = I_2 - \frac{1}{2} + \frac{1}{4m}\bar{Q}^\alpha Q_\alpha , \tag{3}$$

where a summation over α and β from 1 to 2 is implied; $T_\pm = T_3 \pm iT_1 \cdot Y^n$ and T_a commute with Q_α and $\bar{Q}_\alpha$ and with each other $[Y^n, T_a] = 0$; and each satisfies the relations of the SU(2) algebra for the angular momentum and the isospin, respectively. By virtue of the latter relation, the Casimir operators $\sum_{n=1}^{3}(Y^n)^2$ and $\sum_{a=1}^{3}(T_a)^2$ have the eigenvalues $Y(Y+1)$ and $T(T+1)$, respectively, for irreducible representations. The quantities Y and T, which take on integer and half-integer values, determine the superspin and superisospin of the irreducible representation.

Let us examine the content of the irreducible representations with $P^2 = Z^2$. Since the operators Q_α, Y^3, and T_2 commute, we can find a normalized state $|\Phi\rangle$ which, being a Clifford vacuum,

$$Q_\alpha |\Phi\rangle = 0, \tag{4}$$

simultaneously has the maximum eigenvalues of the operators Y^3 and T_2, which are Y and T, respectively:

$$Y^3|\Phi\rangle = Y|\Phi\rangle = 0; \quad Y_+|\Phi\rangle = 0; \quad T_2|\Phi\rangle = T|\Phi\rangle ; \quad T_+|\Phi\rangle = 0, \tag{5}$$

where $Y_\pm = Y^1 \pm iY^2$. It then follows from (2.1)–(5) that the superspin and the superisospin of the state $|\Phi\rangle$ are Y and T and that in the basis with the quantum numbers $|S, S_3; I, I_2\rangle$, where S and S_3 are the spin and its third projection, and I and I_2 are the isospin and its second projection, this state is

$$|\Phi> = |Y, Y; \ T + 1/2, \ T + 1/2 >.$$

For given values of the superspin Y and of the superisospin T, we can work from the $|\Phi\rangle$ state, using the operators $\bar{Q}_\alpha$, $M_- = M^1 - iM^2$, and I_-, to construct all four states with definite values of the spin and the isospin and different spin and isospin projections S_3 and I_2:

$$|Y, S_3; T + \frac{1}{2}, I_2> = \left(\frac{(Y+S_3)!\,(T + I_2 + {}^1/_2)!}{(2Y)!\,(Y-S_3)!\,(2T+1)!\,(T - I_2 + \frac{1}{2})!}\right)^{1/2}$$

$$\times (M_-)^{Y-S_3}(I_-)^{T-I_2+{}^1/_2}|\Phi>;$$

$$|Y, S_3; T - \frac{1}{2}, I_2> = \left(\frac{2T}{T+I_2+\frac{1}{2}}\right)^{1/2}$$

$$\times\left[\left(\frac{T - I_2 + \frac{1}{2}}{2T+1}\right)^{1/2} |Y, S_3; T + \frac{1}{2}, I_2> +\right.$$

$$+ \frac{i}{4m}\left(\frac{2T+1}{T+I_2+\frac{3}{2}}\right)^{1/2}$$

$$\left.\times\ \bar{Q}^\alpha\bar{Q}_\alpha\,|Y, S_3; T + \frac{1}{2}, I_2 + 1>\right];$$

$$|Y + \frac{1}{2}, S_3; T, \ I_2> = \left(\frac{2T+1}{2m\,(2Y+1)\,(T+I_2+1)}\right)^{1/2}$$

$$\times[(Y + S_3 + \frac{1}{2})^{1/2}\ \bar{Q}^1\,|Y, S_3 - \frac{1}{2};$$

$$T + \frac{1}{2}, I_2 + \frac{1}{2}> + (Y - S_3 + \frac{1}{2})^{1/2}\ \bar{Q}^2\,|Y, S_3 + \frac{1}{2}; T + \frac{1}{2}, I_2 + \frac{1}{2}>];$$

$$|Y-\frac{1}{2}, S_3; T, I_2> = \left(\frac{2T+1}{2m(2Y+1)(T+I_2+1)}\right)^{1/2}$$

$$\times[(Y-S_3+\frac{1}{2})^{1/2} \bar{Q}^1 |Y, S_3-\frac{1}{2};$$

$$T+\frac{1}{2}, I_2+\frac{1}{2}> - (Y+S_3+\frac{1}{2})^{1/2} \bar{Q}^2 | Y, S_3+\frac{1}{2}; T+\frac{1}{2}, I_2+\frac{1}{2}>].$$

In this manner we can find all possible matrix elements of the spinor generators Q_α and $\bar{Q}^\alpha$.

In general, therefore, the irreducible representation with $P^2 = Z^2$, with superspin Y and superisospin T, has the following spin-isospin (S, I) structure:

$$(Y, T+\frac{1}{2}) \oplus (Y, T-\frac{1}{2}) \oplus (Y+\frac{1}{2}, T) \oplus (Y-\frac{1}{2}, T). \tag{6}$$

Two series of representations with a content sparser than that in the general case, (7), have a common lower representation: a Fayet-Sohnius hypermultiplet corresponding to the values $Y = T = 0$ and the structure $(1/2,0) \oplus (0,1/2)$. These two series are (1) $Y=0$, with $(1/2, T) \oplus (0, T+1/2) \oplus (0, T-1/2)$, and (2) $T=0$, with $(Y+1/2,0) \oplus (Y,1/2) \oplus (Y, -1/2,0)$.

For the $N=2$ versions of expanded supergravity theories and super–Yang-Mills representations, the following representations are of interest: $Y=3/2$, $T=0$, with $(2,0) \oplus (3/2,1/2) \oplus (1,0)$, and $Y=1/2$, $T=0$, with $(1,0) \oplus (1/2,1/2) \oplus (0,0)$.

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