Stochastic quantization of gauge theories

A. M. Semikhatov
P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

(Submitted 10 May 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **38**, No. 1, 38–40 (10 July 1983)

The form of the gauge generators of an arbitrary gauge theory with a closed algebra determines a family of diffusion processes for which stochastic quantization is equivalent to quantization in a fixed gauge through a path integral constructed in a Lagrangian formalism with a quantum measure.

PACS numbers: 11.15.Kc

To lay the groundwork for a stochastic quantization of a gauge theory by the method of Parisi and Wu[1] involves constructing a diffusion process, free of a gauge and free of ghosts, which is of such a nature that taking an average of gauge-invariant quantities over its equilibrium distribution is the same as taking an average by means of a path integral containing a gauge and ghosts. This correspondence has been established for a well-chosen diffusion process and also for a special class of closed gauge algebra.[1,2]

In this paper we do the following: i) We expand the allowed class of gauge algebras. ii) We derive equations which single out, for a given algebra, a family of diffusion processes. iii) For these processes we establish a stochastic quantization, taking into account a quantum measure in a path integral. The allowed diffusion processes arise as a result of the elimination, through the use of the BRST symmetry, the ghost degrees of freedom from a diffusion process matched with this symmetry on the manifold of *all* field involved in the construction of the path integral.

1. Adopting the notation of Ref. 3, we assume $\phi = (\phi^i)$, where $i = 1,...,n$ is the gauge field, and M is the (super-) manifold of all ϕ on which the action $\mathscr{S}(\phi)$ is given. This action is cancelled by the gauge generators $R_\alpha(\partial_r \mathscr{S}/\partial\phi^i R^i_\alpha = 0$, $\alpha = 1,...,m)$, which form a closed algebra with the structure functions $f^\alpha_{\beta\gamma}(\phi)$. We obtain the BRST-symmetry formalism in the following way[3]: We expand the space of fields up to the supermanifold $\mathscr{M}$ with the coordinates $\Phi^A = (\phi^i, x^a) \equiv (\phi^i, c^\alpha, \bar{c}_\alpha, \pi_\alpha)$ with the Grassman parities $\epsilon(c^\alpha) = \epsilon(\bar{c}_\alpha) = \epsilon(\pi_\alpha) + 1 = \epsilon_\alpha + 1, \epsilon_\alpha = \epsilon(R_\alpha)$. For the functions f and h on the supermanifold $\mathscr{N}$ with the independent coordinates $(\Phi^A, \Phi^*_A), \epsilon(\Phi^*_A) = \epsilon_A + 1, \epsilon_A = \epsilon(\Phi^A)$ ($\mathscr{N} = \Pi T^*\mathscr{M}$ is a cotangential stratification with a shifted parity) we define the antibrackets[3]

$$(f, h) = \frac{\partial_r f}{\partial \Phi^A}\,\frac{\partial_l h}{\partial \Phi^*_A} - \frac{\partial_r f}{\partial \Phi^*_A}\,\frac{\partial_l h}{\partial \Phi^A}\,. \tag{1}$$

we will use ab$f$ to denote the operator $h \rightarrow \mathrm{ab}f(h) = (f,h)$.

For the most general method for constructing a path integral[3] in a fixed gauge, the expectation values are

$$<f> = \int_{\mathcal{M}} d\,\Phi \exp(-\Lambda(\Phi)) f(\Phi) \exp(-S_\Psi(\Phi)), \tag{2}$$

where

$$H = \phi_i^* R_\alpha^i c^\alpha + \frac{1}{2} c_\gamma^* f_{\alpha\beta}^\gamma c^\beta c^\alpha (-1)^{\epsilon_\alpha} + \bar{c}^{*\alpha} \pi_\alpha; \quad S_\Psi = \mathcal{S} + (\Psi, H), \tag{3}$$

Ψ is the nondegenerate[3] gauge fermion [$\epsilon(\Psi) = 1$], of the form $\Psi(\Phi) = \bar{c}_\alpha \chi^\alpha(\Phi)$ and $d\Phi \exp(-\Lambda)$ is the volume form (*quantum measure*) on $\mathcal{M}$, which satisfies the equation

$$(\Lambda, H) = \operatorname{div}(\mathrm{ab}H) \equiv \frac{\partial_l}{\partial \Phi^A} (-1)^{\epsilon_A} \frac{\partial_l H}{\partial \Phi_A^*} . \tag{4}$$

2. Going over to a stochastic quantization, we treat expectation values of the type in (2) as average over an equilibrium distribution of the diffusion on $\mathcal{M}$ controlled by[4] the elliptic operator

$$L = \frac{1}{2} g^{-1/2} \exp S_\Psi \partial_A [(-1)^{\epsilon_A} \exp(-S_\Psi) g^{AB} g^{1/2} \partial_B], \tag{5}$$

where $g^{AB} = (-1)^{\epsilon_A \epsilon_B} g^{BA}$, $g = [\mathrm{Ber}(g^{AB})]^{-1}$, $\partial_A = \partial_l / \partial \Phi^A$. The equilibrium distribution density P with respect to the volume element $d\Phi g^{1/2}$ satisfies[4] the steady-state Fokker-Planck equation

$$L^* P = 0, \tag{6}$$

where L^* is the adjoint of L with respect to this volume form. For arbitrary values of $g^{AB}(\Phi)$, a solution of Eq. (6) is $P = \exp(-S_\Psi)$. By virtue of the BRST symmetry, which can be described in the notation in (3) as $(H, S_\Psi) = 0$, this solution is BRST-invariant. The latter assertion is consistent with the following *principle for matching diffusion with the BRST symmetry.*

We require BRST-invariance of the diffusion; i.e., $[L^*, \mathrm{ab}H] = 0$. Then (i) the vector field $\mathrm{ab}H$ is a Killing vector field[5] in the metric $g^{AB}(\Phi)$, generated by the operator (5), and (ii) $\Lambda = \ln g^{-1/2}$ satisfies Eq. (4) by virtue of the Killing nature. It is this solution for the quantum measure which we choose below.

3. We now perform a "dimensional reduction" from $\mathcal{M}$ to the M of the diffusion which we have introduced, by taking an average over all the fields x^a. For this purpose, we assume that the metric $g^{AB}(\Phi)$ can be chosen in a block diagonal form, $g^{ib} = 0$, with the blocks $g^{ij}(\phi)$ and $g^{ab}(\phi, x)$.

(i) Langevian equations can then be written for the diffusion controlled by the operator L:

$$d\phi^i = \sigma^{ij}(\phi) dw^j + b^i(\phi, x) dt, \tag{7a}$$

$$dx^a = \sigma^{ab}(\phi, x) dw^b + b^a(\phi, x) dt, \tag{7b}$$

$$b^A = \frac{1}{2} g^{-1/2} \exp S_\Psi \partial_B [(-1)^{\epsilon_B} \exp(-S_\Psi) g^{1/2} g^{BA}], \tag{7c}$$

where $\sigma^{AC}\sigma^{BC} = g^{AB}$, the differentials are to be understood in the Ito sense,[4] and w^A denotes the components (all of which are of even parity!) of the $(n + 3m)$-dimensional Wiener process.[4] Our goal is therefore to express the average effect of diffusion perturbation (7a) by means of another diffusion process, (7b), and we will do this in terms of a Fokker-Planck equation.

(ii) Since the field ab$H$ is of a Killing nature in the metric g^{AB} on $\mathscr{M}$, each of the generators R_α is a Killing generator in the metric $g^{ij}(\phi)$ on M:

$$g^{ij}{}_{,k} R^k_\alpha - g^{ik}\partial_k R^j{}_\alpha - (-1)^{\epsilon_i \epsilon_j} g^{jk}\partial_k R^i_\alpha = 0. \tag{8}$$

(iii) We have $\Lambda(\Phi) = \Lambda_0(\phi) + \Lambda_1(\phi,x)$, where $\Lambda_0 = \frac{1}{2}\ln(\mathrm{Ber}(g^{ij}))$ and $\Lambda_1(\phi,x)$ does *not* transform through a berezinian under a change of coordinates on M.

If Λ_1 is actually independent of ϕ (i.e., if the quantum measure has a structure in the accuracy of the corresponding direct product), then by substituting (5) and (3) into (6), using (4) for Λ of this type, and making use of the Killing nature of the field ab$H$, we find, taking an average over x^a, that

$$p(\phi) = \int dx \exp(-\Lambda_1(x) - S_\Psi(\Phi)) \tag{9}$$

is the equilibrium probability distribution density with respect to the volume form $d\phi \exp(-\Lambda_0)$ for the diffusion process on M controlled by the operator

$$L^{red} = \frac{1}{2}\exp\mathcal{S}\,\nabla^i(\exp(-\mathcal{S})\nabla_i\,\cdot) + R^i_\alpha\,\omega^\alpha\nabla_i, \tag{10}$$

where ∇_i is a covariant derivative constructed on the metric g^{ij}, and

$$\omega^\alpha = \frac{\int dx\, c^\alpha \partial_A(g^{AB}(-1)^{\epsilon_A}\exp(-\Lambda - S_\Psi)\partial_B\Psi)}{\int dx \exp(-\Lambda - S_\Psi)}. \tag{11}$$

4. It follows from (8) that the generators R_α commute with the first term in (10). Using the standard arguments [which include (8)],[2] we find that in calculating the expectation values of the gauge-invariant quantities $f(\phi)$ we can use the diffusion on M determined by the Langevin equation.

$$d\phi^i = \sigma^{ij}dw^j + g_0^{-1/2}\exp(-\overleftarrow{\mathcal{S}})\,\partial_j((-1)^{\epsilon_j} g^{ji} g_0^{1/2}\exp\overrightarrow{\mathcal{S}})dt, \tag{12}$$

where $\sigma^{ij}\sigma^{kj} = g^{ik}$, $g_0 = (\mathrm{Ber}\, g^{ij})^{-1} = \exp(-2\Lambda_0)$.

For theories in which $R^i_\alpha(\phi) = r^i_\alpha + t^{ij}_\alpha\phi^j$, $\epsilon_i = 0$ where r^i_α and $t^{ij}_\alpha = -t^{ji}_\alpha$ do not depend on ϕ (as is the case in a Lang-Mills theory), Eqs. (8) are satisfied for $g^{ij} = \delta^{ij}$, for example.

5. The conditions for the compatibility of Eq. (8) with Eq. (4) for our choice of Λ are $f^\alpha_{\alpha\beta}(\phi)(-1)^{\epsilon}\alpha = 0$. This is the sole *a priori* restriction on the closed gauge algebra for which a stochastic quantization by means of diffusion (10), (12), with arbitrary $g^{ij}(\phi)$ satisfying conditions (8), can be justified by the method used here. For theories with *local* structure functions $f^\alpha_{\alpha\beta}(-1)^{\epsilon}\alpha$ is proportional to $\delta(0)$ and can be set equal to zero in the framework of a dimensional regularization.

I thank V. Ya. Faĭnberg for interest and useful discussions, and I thank A. A. Tseĭtlin for an introduction to stochastic quantization.

[1]G. Parisi and Y. Wu, Scientia Sinica **24**, 483 (1981).
[2]T. Hori, in: Stochastic Quantization of Gauge Theory and Supergravity, Tokyo Metropolitan Univ. Preprint, 1983; L. Baulieu and D. Zwanziger, Nucl. Phys. **B193**, 103 (1981).
[3]I. A. Batalin and G. A. Vilkovisky, Phys. Lett. **102B**, 27 (1981).
[4]H. P. McKean, Stochastic Integrals, Academic, New York, 1969 (Russ. transl. Mir, Moscow, 1972).
[5]S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1, Wiley, New York, 1963 (Russ. transl. Nauka, Moscow, 1982).

Translated by Dave Parsons
Edited by S. J. Amoretty