Decay of initial steplike perturbation in the Korteweg–de Vries equation

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Exact formulas are obtained with which to determine the time evolution of a perturbation by the methods of the inverse scattering problem. The asymptotic form of the solution in the vicinity of the wave function is determined for large values of the time.

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As is well known, processes in nondissipative nonlinear media with weak dispersion are described by the Korteweg–de Vries (KdV) equation, the reduced form of which is $u_t + 6uu_x + u_{xxx} = 0$. In the case of initial perturbations $u(x,0)$ that decrease rapidly as $x \to \pm\infty$, a remarkable procedure was discovered in[1] for integrating this equation, by using the inverse scattering problem. It is of interest to study the time evolution of initial perturbations $u(x,0)$ of the step-function type, i.e., $u(x,0) \to -c^2 (c>0)$ as $x \to -\infty$ and $u(x,0) \to 0$ as $x \to +\infty$. This problem was first considered in[2,3], where the Witham approximate (quasiclassical) method was used to find the asymptotic solution as $t \to +\infty$. In the present article this problem is solved using the ideas of[1], namely, exact formulas are obtained for a temporal transformation of the scattering data, and make it possible to obtain the solution $u(x,t)$ for arbitrary t with the aid of the linear integral equations of scattering theory, while the asymptotic form of $u(x,t)$ as $t \to +\infty$ is obtained in the region $x > 4c^2 t - (2c)^{-1} \ln t^N$, which contains the wave front.

We consider the Schrödinger equation

$$-y'' + vy = k^2 y \tag{1}$$

with a potential $v = v(x)$ that tends sufficiently rapidly to zero as $x \to -\infty$ and to c^2 as $x \to +\infty$. Let $\psi_1(x,k)$ and $\psi_2(x,k)$ be the solutions of (1) with asymptotic forms

$$\psi_1(x,k) \sim \begin{cases} \exp[ikx] + S_{21}(k)\exp[-ikx], & x \to -\infty \\ S_{22}(k)\exp[ik_1 x], & x \to +\infty \end{cases}$$

$$\psi_2(x,k) \sim \begin{cases} S_{11}(k)\exp[-ik_1 x], & x \to -\infty \\ \exp[-ik_1 x] + S_{12}(k)\exp[ik_1 x], & x \to +\infty, \quad (|k| > c), \end{cases}$$

where $\mathrm{Im}\,k = 0$, $k_1 = \sqrt{k^2 - c^2}$, and we choose the branch of the root in the k plane with cut $[-c, c]$ and with the condition $k_1 > 0$ at $k > c$. The scattering matrix $\|S_{ij}(k)\|$ has definite properties which make it possible to reconstruct it from the coefficient $S_{21}(k)$ and the eigenvalue of (1). Let $-\kappa_l^2$ $(l = 1, 2, \ldots, m)$ be the eigenvalues of (1), and let $m_l^{(1,2)} = \|f_l^{(1,2)}\|^{-2}$, where $f_l^{(1,2)}(x)$ are eigenfunctions that are fixed by the conditions at infinity, namely, $f_l^{(1)}(x) \sim \exp[-\sqrt{\kappa_l^2 + c^2}\, x]$ $(x \to +\infty)$ and $f_l^{(2)}(x) \sim \exp[\kappa_l x]$ $(x \to -\infty)$.

These characteristics are the initial data for the inverse scattering problem, and they can be used to reconstruct the potential $v(x)$. The corresponding formalism was developed in[4] for the case $v(x) \to +c^2$ $(x \to -\infty)$ and $v(x) \to 0$ $(x \to +\infty)$. The procedure of that paper can be applied without modification to the case considered here. Namely, the potential $v(x)$ can be determined from any of the formulas

$$v(x) = c^2 - 2\frac{d}{dx}A_1(x,x) = 2\frac{d}{dx}A_2(x,x), \tag{2}$$

where $A_1(x,y)$ and $A_2(x,y)$, as functions of y (x is a parameter), satisfy the integral equations

$$A_1(x,y) + \int_x^\infty \Omega_1(\eta + y)A_1(x,\eta)\,d\eta + \Omega_1(x+y) = 0, \qquad (y > x), \tag{3}$$

$$A_2(x,y) + \int_{-\infty}^x \Omega_2(\eta + y)A_2(x,\eta)\,d\eta + \Omega_2(x+y) = 0, \qquad (y < x) \tag{4}$$

with kernels

$$\Omega_1(z) = \sum_{l=1}^m m_l^{(1)} \exp[-\sqrt{\kappa_l^2 + c^2}\, z] + \frac{1}{2\pi}\int_0^c |S_{22}(k)|^2 \times \exp[-\sqrt{c^2 - k^2}\, z]\,dk + \frac{1}{2\pi}\int_{-\infty}^\infty S_{12}(k)\exp[ik_1 z]\,dk_1, \tag{3'}$$

$$\Omega_2(z) = \sum_{l=1}^m m_l^{(2)} \exp[\kappa_l z] + \frac{1}{2\pi}\int_{-\infty}^\infty S_{21}(k)\exp[-ikz]\,dk. \tag{4'}$$

We assume now that the potential $v = v(x)$ depends on the time and satisfies the KdV equation. Then, reasoning essentially in the same manner as in[1], we can show that the scattering data vary with time in accordance with the law

$$S_{21}(k,t) = S_{21}(k)\exp[-8ik^3 t]; \quad S_{12}(k,t) = S_{12}(k)\exp[ik_1(8k^2 + 4c^2)t] \quad (|k| > c);$$

$$S_{22}(k,t) = S_{22}(k)\exp[-\sqrt{c^2 - k^2}(4k^2 + 2c^2)t - 4ik^3 t] \quad (|k| < c);$$

$$\kappa_l(t) = \kappa_l; \quad m_l^{(2)}(t) = m_l^{(2)}\exp[-8\kappa_l^3 t]; \tag{5}$$

$$m_l^{(1)}(t) = m_l^{(1)}\exp[\sqrt{\kappa_l^2 + c^2}(8\kappa_l^2 - 4c^2)t].$$

… (2)–(5) enable us to find the solution $v(x,t)$ of the KdV equation with initial conditions $v(x,0)$ that have the asymptotic form $v(x,0)\to 0$ $(x\to -\infty)$ and $v(x,0)\to +c^2$ $(x\to +\infty)$. If we put $v(x,0)=u(x,0)+c^2$, then the solution of the initial problem can be determined from $v(x,t)$ with the aid of the transformation $u(x,t)=v(x-6c^2t,t)-c^2$. As a result we obtain for $u(x,t)$, in obvious fashion, formulas analogous to (2), and the integral equations (3) and (4) with kernels $\Omega_{1,2}(z,t)$ that depend explicitly on the time. Equation (3) turns out to be more convenient for the study of the asymptotic form of $u(x,t)$ as $t\to\infty$ and at large positive x. We consider the case when Eq. (1) does not have a discrete spectrum $(m=0)$. Let $x>4c^2t-(2c)^{-1}\ln t^{n+1}$, where $n>0$ is any integer. In this region, the kernel $\Omega_1(z,t)$ is approximated sufficiently well by a finite number of terms $(\geqslant n)$ of its asymptotic form as $t\to\infty$, so that the problem can be reduced to a solution of the integral equation (3) with a degenerate kernel. As a result, after making the appropriate estimates, we obtain the following formula for $u(x,t)$ in the region $x>4c^2t-(2c)^{-1}\ln t^{n+1}$:

$$u(x,t)=-2\frac{d^2}{dx^2}\ln\det\{E+A(x,t)\}+o(t^{-1/4}), \qquad (6)$$

where E is a unit matrix of order n, and $A(x,t)$ is a matrix with elements

$$A_{ik}(x,t)=\sqrt{\frac{2}{\pi}}\left|S^{\bullet}_{22}(0)\right|^2 c^{2+i-k}\sum_{j=0}^{n-i}\frac{(2i+2j)!}{(i-1)!\,j!\,(i+j)!}$$

$$\times(64c^3t)^{-i-j-\frac{1}{2}}\int_{cx-4c^3t}^{\infty}\xi\exp[-2\xi]\,d\xi.$$

An analysis of formula (6) shows that in this region the solution $u(x,t)$ decays as $t\to\infty$ into $[(n+1)/2]$ solitons, the fastest of which is of the form $(x>4c^2t-(2c)^{-1}\ln t^3)$

$$u(x,t)\sim -2c^2\,\mathrm{ch}^{-2}\left(cx-4c^3t+\frac{1}{2}\ln\frac{256\sqrt{2\pi}(c^3t)^{3/2}}{|S'_{22}(0)|^2c^2}\right).$$

The distance between the neighboring solitons increases like $\ln^{1/c}$, in agreement with[2,3].

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