Dynamics of collapse of electromagnetic solitons

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We obtain self-similar solutions for the collapse of disk-shaped electromagnetic formations for the case $\epsilon \lesssim 0$.

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It is known that the character of the penetration of small-amplitude electro-magnetic waves ($\omega^2 = \omega_{pe}^2 + k^2c^2$) into a plasma is determined by the sign of the dielectric constant $\epsilon_0 = 1 - \omega_{pe}^2/\omega^2$. If ϵ_0 is positive, then the plasma is transparent to waves of given frequency, and the depth of penetration of the radiation is determined by the dissipative processes. In the opposite case, the wave penetrates into the plasma only to the depth of the skin layer $\mu^{-1} = c/\sqrt{\omega_{pe}^2 - \omega^2}$. When the amplitude is increased to the level $W/nT > (\Delta kc/\omega_{pe})^2$ (Δk is the width of the spectrum), the picture becomes more complicated, so that an important role is assumed by the radiation-pressure force $F = -\nabla U$ (where $U = ne^2|E|^2/4m\omega^2$ [1]), which leads to perturbation of the density by an amount $\delta n \sim -n_0(W_0/nT)$, and the wave can penetrate into the plasma even if the dielectric constant is negative. The solution of the one-dimensional boundary-value problem then leads to the conclusion that if ϵ_0 is small enough, then electrosonic solitons, i.e., stable local density minima of width $\mu^{-1} \sim \delta$, filled with electromagnetic radiation, propagate into the interior of the plasma.[2] Under real conditions these solitons are bounded in the other two spatial coordinates, so that in the case of interest to us, that of an axially-symmetrical formation (z, R) (the z axis is directed along the group velocity), the parameter δ/R_0 (R_0 is the characteristic radius) should be regarded as finite. However, three-dimensional formations of the soliton type are, generally speaking, unstable with respect to self-compression at $W/nT > (\Delta kc/\omega_{pe})^2$, i.e., they collapse[3] (this phenomenon is analogous to the instability of cold Langmuir gas, discovered in 1964 by Vedenov and Rudakov[4] and investigated in the nonlinear stage by Zakharov[5]). Assuming $\delta/R_0 \ll 1$ (disk), let us consider the dynamics of such a collapse.

The fundamental system of equations, in terms of the variables $t \to \omega t$ and $x \to \omega x/c$, is of the form[2]

$$2i\frac{\partial \mathbf{E}}{\partial t} + \nabla^2\mathbf{E} + \epsilon_0\mathbf{E} - \delta n\mathbf{E} = 0 \tag{1}$$

and

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right)\delta n = \nabla^2|E|^2 \tag{2}$$

where

$$E^2 \to \frac{E^2}{16\pi nT}\frac{\omega_{pe}^2}{\omega^2}, \quad c_s \to \frac{c_s}{c} \equiv c_0$$

and the terms $\partial^2 E/\partial t^2$ and $\nabla \operatorname{div} E$, which are small in our case, have been left out from (1). The system (1) and (2) has the following subsonic particular solutions: if $\epsilon_0 = 0$ ($\omega^2 = \omega_{pe}^2$) and $\partial^2\delta n/\partial t^2 \ll c_0^2\nabla^2\delta n$ (the "adiabatic limit"), then we have in a reference frame that moves together with the soliton[6,7]:

$$E = E_0\left(\mathrm{ch}\frac{E_0}{\sqrt{2}}z\right)^{-1}\exp\left[i\frac{E_0^2}{4}t + iS\right], \tag{3}$$

where S is an arbitrary phase.

If $\epsilon_0 < 0$ ($|\epsilon_0| \ll 1$), then the width of the soliton along the z axis is fixed (μ^{-1}). Choosing $E = E(\sqrt{|\epsilon_0|}(z - Mc_0t))$ (where $M = v_g/c_s$ is the Mach number), we obtain

$$E = E_0(\mathrm{ch}\sqrt{|\epsilon_0|}(z - Mc_0t))^{-1}\exp[iMc_0(z - Mc_0t) + iS], \tag{4}$$

$$2|\epsilon_0| = E_0^2(1 - M^2)^{-1}. \tag{5}$$

We seek the solution of the system (1) and (2) for a three-dimensional formation of a disk in the form (3) or (4), assuming E_0 and S to be slowly varying functions of R and t ($E'_{0R}/E_0 \ll 1$).

For the case $\epsilon_0 = 0$, we substitute the solution (3) into the system (1) and (2), separate the real and imaginary parts, and integrate with respect to z with allowance for the relation $\int_{-\infty}^{+\infty} E_0[\cosh(E_0/\sqrt{2})z]^{-1}dz = \sqrt{2}\pi$. We obtain a simple system of equations consisting of the continuity equations for E_0 and a nonlinear equation for the phase $S(R,t)$, which has a self-similar solution in terms of the variables $\xi = R/\tau^{1/3}$ and $\tau = t_0 - t$ (where t_0 is the collapse time). In terms of the variables R and t, this solution takes the form

$$\tilde{E}_0(R,t) = \frac{\sqrt{2}}{3(t_0 - t)^{1/3}}\left[1 - \frac{R^2}{(t_0 - t)^{2/3}}\right]^{1/2}\exp\frac{i}{3(t_0 - t)^{1/3}} \times \left(1 - \frac{R^2}{2(t_0 - t)^{2/3}}\right)E_0 e^{iS}. \tag{6}$$

The thickness of the disk is

$$\delta \sim (t_0 - t)^{1/3}\left(1 - \frac{R^2}{(t_0 - t)^{2/3}}\right)^{-1/2},$$

since E_0 is approximately constant. Near the point $\xi \geqslant 1$, the solution (6) does not hold, since $\delta/R_0 \to \infty$ and $\partial E/\partial R$ has a discontinuity. In this region we have

$$\tilde{E}_0 = \frac{1}{R^2}\exp\left(-\frac{i}{3R}\right). \tag{7}$$

It follows from (7) that the fraction of the energy in the region $\xi \geqslant 1$ is small in comparison with the energy of the collapsing formation, and it can be assumed that the region of large ξ does not influence the solution (6), For the case $\epsilon_0 < 0$ ($|\epsilon_0| \ll 1$) it is necessary to introduce a new function $\mathcal{E} = E\exp i(\epsilon_0/2)t/\sqrt{1 - M^2}$, and then the system of equations for $\mathcal{E}$ will coincide in form with the system (1,2) for the function E at $\epsilon_0=0$. We seek the solution of this system in the form $\mathcal{E} = \phi \exp iS$ (where $\phi = E_0(R,t)[\cosh\sqrt{|\epsilon_0|}\,(z - Mc_0t)]^{-1} \exp iMc_0(z - Mc_0t)$ and $S(R,t) = c(R,t) - 2\ln(t_0 - t)$, separating the real and imaginary parts, and integrating with respect to z with allowance for the relations

$$\int_{-\infty}^{+\infty} E_0(\cosh\sqrt{|\epsilon_0|}\,\xi)^{-1} d\xi = \pi \frac{E_0}{\sqrt{|\epsilon_0|}} \text{ and } \int_{-\infty}^{+\infty} E_0^3 (\cosh\sqrt{|\epsilon_0|}\,\xi)^{-3} d\xi = \frac{\pi}{2} \frac{E_0^3}{\sqrt{|\epsilon_0|}}$$

we obtain a relatively simple system of equations having a self-similar solution in terms of the variables $\eta = R/\sqrt{\tau}$ and $\tau = t_0 - t$. If it is assumed that the term $\nabla^2 E_0/2E_0$ is small in accordance with the condition $E'_R/E \ll 1$, then the solution can be written in the form

$$\tilde{E}_0(R,t) = \frac{2}{\sqrt{t_0 - t}} \left[1 - \frac{R^2}{8(t_0 - t)}\right]^{1/2} \exp\left[-i \frac{R^2}{4(t_0 - t)}\right] \equiv E_0 e^{ic}, \qquad (8)$$

At $R \gtrsim \sqrt{8(t_0 - t)}$ expression (8) does not hold, since $\partial E/\partial R$ has a discontinuity. In this case it is necessary to take $\nabla^2 E$ into account and solve more exact equations.

The solution (8) and relation (5) show that a disk initially flat at $\epsilon_0 < 0$ and $|\epsilon_0| \ll 1$ will be deformed in the course of the collapse and be converted into a cone with a "blunted apex" pointing in a direction opposite to the motion. The cone profile $z(R,t)$ at $t \to t_0$ can be easily obtained. We have $z(R \to R_0) \sim R$ and $z(R \to 0) \sim \text{const}$. As to the radiation of sound, by choosing the initial field amplitude it is always possible to make the collapse time large, and consequently all the accelerations ($\sim c_s/t_0$) can be made small. Therefore the conditions under which sound radiation can be neglected are realistic.

It is interesting to note that, owing to the collapse, the depth of penetration of a soliton-type electromagnetic wave is finite at $\epsilon_0 < 0$ and cannot exceed $c_s t_0$, so that the energy is dissipated near the boundary.

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