Certain spaces that violate the causality principle

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Exact solutions, particular cases of which are the solutions of Gödel and Van Stockum, were obtained for Einstein's equation with a cosmological term inside rotating matter without pressure.

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1. It is known that certain solutions of Einstein's equation admit of the existence of closed timelike or isotropic (causal) curves.[1] This contridicts the usual concepts of causality, and therefore one speaks of such solutions as violating causality. The violation of causality is called trivial if it can be avoided by using only global structure of space. In all other cases, which are called nontrivial, closed causal curves cannot be eliminated without changing the local properties of space.

The first solution of Einstein's equations (without a cosmological term), in which the causality principle is violated in nontrivial fashion, was obtained by Van Stockum,[2] who considered the gravitational field of a rotating dust (pressure-free) cylinder. Later on, Gödel[3] constructed a solution of Einstein's equation with a cosmological term for a rotating dust cloud of constant density, which also violated causality in nontrivial fashion. The structures of the solutions of Van Stockum inside the cylinder and of Gödel's solutions are surprisingly similar, which suggests that there can exist a certain solution, particular cases of which are the two aforementioned spaces. In this article we present a solution obtained by us for Einstein's equation with a cosmological term, which is a generalization of the solutions of[2,3].

2. We consider the metric[1)]

$$ds^2 = dt^2 + \left[2C_1^2\left(\int^r d\xi \frac{f_1}{f_2}\right)^2 - f_1^2\right]d\phi^2 + 2\sqrt{2}\,C_1\left(\int^r d\xi \frac{f_1}{f_2}\right) \times d\phi\, dt - dr^2 - f_2^2 dz^2. \tag{1}$$

It can be verified by direct calculation that this metric satisfies the Einstein equation

$$R_{ik} - \frac{1}{2} g_{ik} R = \rho u_i u_k - \Lambda g_{ik}, \tag{2}$$

(where $\rho/8\pi k$ is the density of matter ($\rho \geqslant 0$), u^i is the vector of velocity of matter, and Λ is the cosmological constant) in the rest system of the matter if the following relations are satisfied

$$\frac{f_1''}{f_1} + \frac{f_2''}{f_2} = 2\Lambda; \quad f_2\left(\frac{1}{2}\rho + \Lambda\right)^{1/2} = C_1, \quad f_1 f_2\left(\Lambda - \frac{1}{2}\rho\right) = (f_1 f_2')'; \quad 2\Lambda f_1 f_2 = (f_1' f_2)', \tag{3}$$

where $f_1 = f_1(r)$, $f_2 = f_2(r)$, and C_1 is an arbitrary constant. We consider first spaces with constant density. A solution of the Gödel type is obtained by putting $f_2 = \text{const}$, which without loss of generality can be assumed equal to unity. It then follows from (3) that $C_1^2 = \rho$, $\Lambda = \rho/2$, $f_1 = a \sinh\sqrt{\rho}(r + r_0)$, and a and r_0 are constants. The integration constant A in (1) and r_0 are determined from the condition $g_{\phi\phi}(0) = g_{t\phi}(0) = 0$, $A = 1$, and $r_0 = 0$. From this we obtain ultimately

$$ds^2 = dt^2 + a^2(\operatorname{ch}\sqrt{\rho}r - 1)(\operatorname{ch}\sqrt{\rho}r - 3)d\phi^2 + 2\sqrt{2}(\operatorname{ch}\sqrt{\rho}r - 1)d\phi dt - dr^2 - dz^2. \tag{4}$$

For $\sqrt{\rho} = 2/a$, expression (4) can be recast, by changing scale, in a form coinciding with the Gödel solution. Another space with constant density is obtained at $C_1 = 0$. In this case, however, the matter does not rotate and therefore, according to Hawking,[4] the space is causally stable.

3. Let $f_2 \neq \text{const}$. Then, solving the system (3) and stipulating that $g_{\phi\phi}(0)$ and $g_{\phi t}(0) = 0$, we obtain

$$ds^2 = dt^2 + C_2^2\left[2C_1^2(\ln Af_2)^2 + C_1^2 \ln Af_2 + \frac{C_3}{A^2}(1 - A^2 f_2^2)\right]d\phi^2 + 2\sqrt{2}\,C_1 C_2(\ln Af_2)\,d\phi\, dt - dr^2 - f_2^2\, dz^2,$$

where f_2 is determined from the equation

$$r = \pm\int_0^{f_2} \frac{dy}{\left[\frac{C_3}{A^2}(A^2y^2 - 1) - C_1^2 \ln Ay\right]^{1/2}}; \quad \rho = \frac{2C_1^2}{f_2^2} - 2\Lambda; \quad C_3 = \Lambda; \quad f_2(0) = 1/A \tag{5}$$

while C_1, C_2, C_3, and A are arbitrary constants. At $\Lambda = C_3 = 0$, Eq. (5) yields a solution of the Van Stockum type. The integral in (5) can be evaluated for this case by introducing a new radial variable $\tilde{r}$ in accordance with the formula

$$d\tilde{r} = dr/f_2(r).$$

As a result we have

$$ds^2 = dt^2 + \frac{C_1^4 C_2^2}{4}\left(\frac{C_1^2}{2}\tilde{r}^4 - \tilde{r}^2\right)d\phi^2 - \frac{\sqrt{2}}{2} C_1^3 C_2 \tilde{r}^2 d\phi\, dt - \left[\exp - 2\left(\ln A + \frac{C_1^2}{4}\tilde{r}^2\right)\right] \times (d\tilde{r}^2 + dz^2), \quad \rho = 2C_1^2 \exp 2\left(\ln A + \frac{C_1^2}{4}\tilde{r}^2\right). \tag{6}$$

Expression (6) goes over into the Van Stockum solution at $A = 1$ and $C_2 = 2/C_1^2$.

4. Let now $\Lambda \neq 0$. We introduce in (5) a new radial variable $\tilde{r}$

$$d\tilde{r} = dr \frac{\left[\frac{\Lambda}{A^2}(A^2 f_2^2 - 1) - C_1^2 \ln A f_2\right]^{1/2}}{f_2 (a \ln A f_2)^{1/2}}, \qquad a = 2\Lambda - C_1^2 A^2 \tag{7}$$

Then the solution (5) takes the form

$$f_2 = \frac{1}{A} e^{\frac{a\tilde{r}^2}{4}}; \qquad \rho = 2A^2 C_1^2 e^{-\frac{a\tilde{r}^2}{2}} - 2\Lambda. \tag{8}$$

We consider first $a<0$. This case is qualitatively fully equivalent to the Van Stockum solution: the density increases exponentially with $\tilde{r}$ and the closed curves $\tilde{r} = \text{const}$, $z = \text{const}$, and $t = \text{const}$ become timelike starting with a certain $\tilde{r}_1$. Greater interest attaches to the solutions with $a>0$. In this case the density decreases exponentially with $\tilde{r}$, so that a matter boundary $\tilde{r} = \tilde{r}_b$ arises, determined from the condition $\rho = 0$, beyond which the solution (8) loses its physical meaning. We proceed, finally, to the question of causality. Closed timelike curves appear at $1.7 \lesssim C_1^2 A^2/\Lambda < 2$, starting with a certain $\tilde{r}_2$ smaller than $\tilde{r}_b$. At $1 < C_1^2 A^2/\Lambda \lesssim 1.7$ there are no closed causal trajectories inside rotating dust.

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[1)]The metric (1) was obtained by the author with the aid of some heuristic device which he hopes to describe in detail elsewhere.

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