Classical spin and Grassmann algebra

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A Hamiltonian approach to classical dynamics of a particle with spin is formulated. The analog of the functions on phase space in this case are elements of the Grassmann algebra.

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In view of the introduction of transformation groups with anticommuting parameters into the theory of elementary particles,[1] and the intensive discussion of "supersymmetry" (see, for example, Zumino's review[2]), universal interest has been advanced in classical anticommuting quantities, i.e., in the Grassmann-algebra formalism. It appears that the first hint of the possibility of using Grassmann algebra in quantum-mechanics is contained in[3]. The Grassmann algebra turned out to be quite useful in the second-quantization method,[4] where analogs were used of such classical-analysis concepts as differentiation and integration. A corresponding generalization of the theory of Lie groups was also found.[5]

The subject of the present paper is a generalization of classical mechanics, consisting in the fact that phase space contains both ordinary commuting generators as well as anticommuting ones. This generalization has made it possible to construct a classical Hamiltonian mechanics of spin for a pointlike particle. The quantization of this mechanics leads to the usual Pauli-Dirac quantum theory. As is well known, earlier attempts to formulate Hamiltonian mechanics for particles with spin encountered serious difficulties,[6] which have not yet been overcome.[7]

In nonrelativistic theory, we shall describe spin by a three-dimensional vector ζ_k with anticommuting components

$$\zeta_k \zeta_l + \zeta_l \zeta_k = 0, \qquad k, l = 1, 2, 3 \tag{1}$$

(in particular, $\zeta_k^2 = 0$). In other words, we shall assume that the dynamic variables, i.e., functions on phase space, are elements of a Grassmann algebra with three generators, G_3. We introduce complex conjugation, i.e., we consider Grassmann algebra with involution (see the book[4], p. 76) and assume ζ_k to be real. (We recall that in the case of involution the order of the factors is reversed.) Let the classical action be an even real element of G_3. It then takes the form

$$A_\zeta = \int_{t_1}^{t_2} [i \zeta_k \dot{\zeta}_k - H(\zeta)] dt, \tag{2}$$

where $\dot{\zeta}_k \equiv d\zeta_k/dt$ and $H(\zeta)$ is the Hamilton function. Any element of G_3 can be represented as the polynomial of degree not higher than 3. Therefore, the most general Hamiltonian takes the form $H(\zeta)=i\epsilon_{klm}\zeta_k\zeta_l b_m$, where b_m is a real vector. The resultant equations of motion are

$$\dot{\zeta}_k = \frac{1}{2i}\frac{\partial H}{\partial \zeta_k} = \epsilon_{klm}\zeta_l b_m. \tag{3}$$

The solution of these equations describes the precession of the vector ζ_k, and therefore a system with the indicated Hamiltonian is interpreted as a statement of the problem of the motion of spin in a magnetic field.

In accordance with (3), we define the Poisson brackets $\{F(\zeta), G(\zeta)\} \equiv (2i)^{-1}(\vec{\partial}_k F)(G\overleftarrow{\partial}_k)$, where $\vec{\partial}_k$ and $\overleftarrow{\partial}_k$ are the left-hand and right-hand derivatives with respect to ζ_k. In particular,

$$\{\zeta_k, \zeta_l\} = \frac{1}{2i}\delta_{kl}. \tag{4}$$

The vector of the angular momentum connected with the spin (i.e., the generator of the group of three-dimensional rotations) takes the form

$$S_k = -i\epsilon_{klm}\zeta_l\zeta_m, \qquad \{S_k, \zeta_l\} = -\epsilon_{klm}\zeta_m. \tag{5}$$

The phase space of a particle with spin is constructed by adding to the ordinary six-dimensional space (q_k, p_k) a three-dimensional Grassmann space. The total action can contain the spin-orbit interaction and takes the form

$$A = \int_{t_1}^{t_2}[p_k\dot{q}_k - \frac{1}{2}p_k^2 - V_0(q) + i\zeta_k\dot{\zeta}_k - V_1(q)L_k S_k]\,dt, \tag{6}$$

where $L_k=\epsilon_{klm}q_l p_m$ is the orbital angular momentum and $V_0(q)$ and $V_1(q)$ are the potential functions. Obviously, formula (6) yields the most general form of the local Hamiltonian.

Canonical quantization is carried out by replacing the fundamental Poisson brackets (4) by the anticommutator:

$$[\hat{\zeta}_k, \hat{\zeta}_l]_+ = \frac{1}{2}\hbar\delta_{kl}. \tag{7}$$

Thus, upon quantization the Grassmann algebra goes over into the Clifford algebra. As is well known, the only irreducible representation of the Clifford algebra with three generators is two-dimensional, and we arrive at two-component spinors. Putting $\hat{\zeta}_k=\frac{1}{2}\hbar^{1/2}\sigma_k$, we obtain

$$[\sigma_k, \sigma_l]_+ = 2\delta_{kl}, \quad \hat{S}_k = -i\epsilon_{klm}\hat{\zeta}_l\hat{\zeta}_m = \frac{1}{2}\hbar\sigma_k. \tag{8}$$

We note that in the usual approach the starting point is the commutator $[\hat{S}_k, \hat{S}_l]_- = i\hbar\epsilon_{klm}S_m$, and a simple form of the anticommutator appears only in a representation with spin 1/2. Here the logic is reversed: we postulate the anticommutator (7), and therefore only the spinor representation appears.

The canonical formulation of classical mechanics of spin enables us also to construct a quantum theory on the basis of a continual integral.

To construct a relativistic theory, we need a Grassmann algebra with five generators ζ_0, ζ_k, ζ_5. In this case $\zeta_a=(\zeta_0,\zeta_k)$ form a four-dimensional axial vector, and ζ_5 a pseudoscalar. Just as for a spinless particle, the relativistic action corresponds to a degenerate Lagrangian. To fix the gauge it is necessary to have an additional condition, which can be chosen in the form

$$p_a\zeta_a + m\zeta_5 = 0. \tag{9}$$

The quantization leads to a Clifford algebra with five generators, the representation of which is connected with the Dirac matrices. The relation (9) is transformed into the ordinary Dirac equation $(p\gamma_5 + m\gamma_5)\psi = 0$.

Classical mechanics of a relativistic pointlike particle with spin 1/2 is also of interest in connection with the theory of the "spin string."[8]

We note in conclusion that by using the Grassmann phase space we can also introduce internal symmetry at the level of classical mechanics. In particular, if it is assumed that the three generators ζ_k transform like an isovector, then the quantization leads to a pair of particles with the properties of an isodoublet.

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