Percolation properties of a random two-dimensional stationary magnetic field

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The lines of force of a stationary two-dimensional random magnetic field do not give rise to percolation. Percolation appears along thin filaments in the presence of a weak constant field. Turbulent heat conduction with almost two-dimensional motion of the fluid is discussed.

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Interest in magnetic lines of force, defined by the equation

$$d\mathbf{r} = \mathbf{B}(\mathbf{r})\,d\alpha \tag{1}$$

(α is an arbitrary scalar parameter), reviving Faraday's ideas, has arisen in the last 10 years primarily in connection with the theory of controlled thermonuclear reactions, as well as in connection with problems in astrophysics, where the field is frozen into the plasma, and the plasma defines a natural system of coordinates, in three-dimensional space, in which the (pseudo) vector $\mathbf{B}$, instead of the tensor F_{ik} of Minkowski space, is defined.

We shall examine the problem of the properties of magnetic "lines of force." One of the most important questions here is: Do these lines extend over an infinite distance without interruption in any direction? This formulation of the problem is called the percolation formulation. The word "percolation" suggests a flow. We can imagine thin tubes, instead of lines of force, filled with liquid and we can ask the following question: Does this system of tubes permit transferring liquid over an infinite distance? The percolation formulation of the problem supplements the problem of linkage of lines of force developed in detail by Moffat[1] and others.

Percolation along magnetic lines is interesting primarily because charged particles move along these lines along spirals.

The thermal conductivity of a thermonuclear plasma due to diffusion of electrons also depends on the properties of the magnetic lines of force and the surfaces on which they coil.

In particular, Kadomtsev and Pogutse[2] examined a three-dimensional problem, in which a weak random two-dimensional field $\mathbf{b}(b_x, b_y, 0)$ is superimposed on a strong constant field oriented along the z axis, $\mathbf{B}(0,0B_0)$. Diffusion and heat conduction are determined by the distortion of the lines of force, which depends on $\mathbf{b}$. In their paper, the field $\mathbf{b}$ is expressed with the help of the two-dimensional scalar ϕ, i.e., the z component of the vector potential $\mathbf{a}(0,0,\phi)$

$$b_x = \partial\phi / \partial y, \quad b_y = -\,\partial\phi / \partial x. \tag{2}$$

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The lines of force in the x, y plane circumscribe the maximum in ϕ in the counterclockwise direction, and they circumscribe the minimum in ϕ in the clockwise direction.

When ϕ does not depend on time or the third coordinate z, Kadomtsev and Pogutse[2] reduce the problem to the percolation properties of a random function of two variables.

We shall use the notation[1)] and the normalization

$$<\phi> = 0, \quad <\phi^2> = 1. \tag{3}$$

We shall also assume that ϕ is a sufficiently smooth function with sufficiently low correlation at large distances and a Gaussian probability distribution. In spectral language, this corresponds to a Fourier expansion of ϕ with random phases and amplitudes ϕ_k, such that, for example,

$$<\phi_k^2> \sim \exp(-k^2/k_0^2), \quad k > k_0 \; ; \quad <\phi_k^2> \sim (k/k_0)^{2n}, \; n > 0, \; k \ll k_0. \tag{4}$$

A natural assumption, used in Ref. 2, is that regions with $\phi > \epsilon$ (where $\epsilon > 0$), which occupy less than one-half of the entire area, form isolated "islands," along which percolation occurs. We can introduce the quantity $l(\epsilon)$, which characterizes the average size of an island or the average length of an isoline, which bounds the island.

In the two-dimensional problem, when the plane is separated into two types of regions, it is natural to assume that when regions of the first type form isolated islands, then regions of the second type form a continuous ocean. Correspondingly, the conditions $\phi < \epsilon, \epsilon > 0$ determine a single region, along which percolation occurs.[2)] The quantity $\epsilon = 0$ is a critical value. We can introduce the quantity $l(\epsilon)$, which characterizes the average size of an island or the average length of an isoline $\phi = \epsilon$, bounding an island.

The function $l(\epsilon)$ is estimated in Ref. 2 and it is further proposed that an appropriately averaged quantity $l(\epsilon)$ plays the role of an effective free path in the theory of diffusion and heat conduction.[3)]

In the case of plasma devices with a strong longitudinal field $B_z = B_0$, there is no basis for assuming that small perturbations, i.e., **b**, do not depend on z and/or on time and, in this case, the general idea of Ref. 2 is correct. In this paper, we shall examine, however, a highly idealized problem: 1) either the field B_0, oriented along z, is entirely missing or periodic boundary conditions are imposed along the z axis $z \equiv z + 2\pi R$, describing a torus with large radius R with the z axis oriented along the circle of the torus; 2) the two-dimensional field **b** is entirely independent of z and t; and, 3) particles move along lines of force; in addition, we assume that the Larmor radius r of the spiral traced out by the electron is equal to zero and we ignore collisions and other effects that can cause the electron to jump from one line to another.

After this far-reaching idealization, we obtain a result that is exactly opposite to the assertion made in Ref. 2: electron diffusion does not occur in a two-dimensional random field and the coefficient of diffusion is equal to zero.

The significance of the result lies in the fact that the electron orbits in the x, y plane are closed. Let us fix an initial smooth distribution of the electron density n and the electron temperature T. Motion along a closed orbit leads to averaging over the entire orbit, but to nothing else. In this approximation, the average values $\bar{n}$, $\overline{T}$ on different orbits remain different for all time.

Another formulation of the problem consists of examining a layer of finite thickness, for example, $0 < x < a$, in which the random two-dimensional magnetic field being examined is present. There are no currents flowing along z and creating the field $\mathbf{b}$ either to the right or to the left outside this layer.

We fix one value of n_1 to the left ($x = 0$) and another n_2 to the right of the layer ($x = a$). The flux of particles is

$$q_x = D(n_1 - n_2)/a \tag{5}$$

according to the definition of the coefficient of diffusion. However, in the problem under study, as the thickness of the layer a is increased, the fraction of orbits that are open decreases, since the boundaries $x = 0$ and $x = a$ intersect them. For this reason, the particle flux decreases more rapidly than a^{-1}, either as a^{-m} $(m > 1)$ or exponentially as $\exp(-\alpha k_0 a)$. But this means that there is no definite value of D and, in the limit of large a, the effective value of D approaches zero.

A finite (nonzero) value of D is obtained with a finite Larmor radius r.

In the analogous problem of two-dimensional, rotational, divergence-free stationary motion of a liquid, the turbulent coefficient of diffusion is finite only if the coefficient of molecular diffusion μ is finite (nonzero). In this case, D is proportional to a fractional power of μ.

The problem of two-dimensional stationary motion is, in this respect, similar to the problem of one-dimensional nonstationary motion.[3] An analogous similarity between caustics with an equal number of variables (for example, x and t or x and y) was noted in Ref. 4.

In a truly three-dimensional stationary problem, just as in a two-dimensional nonstationary problem (three variables x, y, z or x, y, t), electron diffusion along magnetic field lines D_e, just as turbulent diffusion in hydrodynamic motion D_t, differs from zero and, in the limit, $\mu = 0$, $r = 0$.

Let us examine, for example, the nonstationary two-dimensional problem with arbitrary discontinuous dependence on time. The particle moves along a closed line of force around a single center within some time τ_1 and moves over a definite distance (not exceeding the orbit even for large τ_1). However, in the next time interval τ_2, the particle moves along a new trajectory that is independent of the first one.

In this case, after several steps, the displacement evidently increases in proportion to the square root of the time, i.e., according to a typical diffusion law.

In the three-dimensional stationary vector field, there are regions of closed lines (Kolmogorov, Arnol'd, and others), but these regions by no means fill the entire space, and there also exists a finite fraction of percolation lines, along which transport occurs from $-\infty$ to $+\infty$ in any direction in a random field.

The general relation between the role of μ, r in the two-dimensional ($2D$) and three-dimensional ($3D$) cases is apparently the same as in the theory of the fast and slow dynamo.[5] The quantities D_e and D_t in the limit $\mu, r \to 0$ depend ($2D$) or do not depend $(3D)$ on μ, r, but in order to construct the complete picture and, especially, to find the gradients, in the smallest scales, the values of r and μ are always important. The dependence of the field on z and t, the influence of μ, r, as well as electron drift in this case are examined in the second part of Ref. 2.

We shall briefly consider the problem of a two-dimensional field, consisting of two parts: a constant field and a random field. The weak random field only distorts the lines of force of the constant field. In separate locations, where the random field is sufficiently strong, islands of closed lines of force, similar to local catastrophes, which are related to the appearance of a local maximum in ϕ and to the saddle point associated with it, are separated. The opposite case, in which a very weak constant field is superimposed on a fixed random field, is interesting.

It is well known that in this case the magnetic flux corresponding to the constant field is compressed into narrow strips (channels), flowing through (creating percolation) the medium, in spite of the tortuousness, in the direction of the constant field. In these channels, the modulus of the field is of the order of the mean square random field; the smallness of the constant field is manifested in the fact that the channels are narrow.

This assertion, which was made by Rosenbluth,[1] sheds light on the fact that the nature of the Fourier spectrum in the case of a random field affects percolation and diffusion.

Let us return to the case of a random field in a strip of finite width. For this strip, the wave vector $k_{\min} = a^{-1}$ corresponds to a field which is essentially indistinguishable from a constant field. If $\phi_k \sim k^m$, then $b_k \sim k^{m+1}$ and we obtain the modulus of the field, constant within the strip, from the condition

$$\bar{b} = (\langle b_k^2 \rangle_{k < a^{-1}})^{1/2} \sim \left(\int_0^{a^{-1}} k^{2m}\, k\, dk\right)^{1/2} \sim a^{-m-2}. \tag{6}$$

Percolation along channels, which do not decrease with increasing a, occurs for $m = -2$. The two-dimensional magnetic field arises in the presence of currents flowing along the z axis; furthermore it is evident that $\Delta\phi = j_z$.

Therefore, $m = -2$ in the long-wavelength part of the spectrum of ϕ corresponds for currents to $(j_z)_k = k^2\phi_k = k^0 = \text{const}$, i.e., to white noise. Thus, it is instructive that random currents, which is often taken to mean precisely uncorrelated currents, i.e., those with the spectrum of white noise, produce, due to the long-range action, average fields which are still accompanied by percolation in the limit $a \to \infty$. The percolation disappears only when $m > -2$, i.e., in the case of anti-correlated currents.

We note finally one more particular case. Let us imagine a constant field in a perfectly conducting liquid. The motion of this liquid amplifies the field. Flux conservation is related here to the fact that with random twisting of the field, the average cosine of the angle between the field and the initial direction approaches zero. As long as the condition for a frozen field is satisfied, the lines of force do not break: percolation remains intact. The picture of narrow channels in this situation arises only over a sufficiently long period of time after the motion begins. In this picture, electrons located near closed lines move only together with the general hydrodynamic motion of the liquid, whereas the characteristic velocity of the electrons in the channels is much higher.

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[1)] The notations b and ϕ differ insignificantly from those used in Ref. 2.

[2)] It has not been ruled out, however, that part of the region with $\phi < \epsilon$ forms lakes inside the islands, which are not related to the percolating ocean.

[3)] We note, however, that the length of an isoline $l(\epsilon)$ for fixed ϵ has a statistical distribution and $l \sim \epsilon^{-2.4}$ is more likely an estimate of the maximum, rather than of the average value of l.

[4)] I am grateful to B. B. Kadomtsev for bringing this point to my attention.

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