Instability of quasineutral electron beam in a semiinfinite metallic waveguide

A. A. Rukhadze and V. Yu. Shafer
Institute of General Physics, Academy of Sciences of the USSR

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The existence of a new type of aperiodic instability of a quasineutral electron beam, which in long systems predominates over the well-known Pierce instability (PI) and has the same current threshold, is demonstrated. The increment slightly above threshold is estimated.

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1. One of the main parameters of any experimental setup used in working with high-current electron beams is the limiting beam current that can be passed through this setup. In this paper, we shall discuss the limiting currents of quasineutral beams,

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whose volume charge in the equilibrium state is neutralized by positive ions. We shall restrict the analysis to electrostatic (potential) instabilities, which are responsible for the limitation of the current of nonrelativistic beams.[1] In this approximation, the characteristic magnetic fields of the beam and the retardation of the potentials are ignored the (velocity of light $c \to \infty$). In addition, in the zeroth-order approximation with respect to the small parameter $(m/M)^{1/3}$ (m and M are the electron and ion masses, respectively), the ions may be assumed to be stationary ($M \to \infty$).

It is well known that in a drift space with finite length L, the current of a neutralized beam is limited by the development of the aperiodic potential Pierce instability[2] with increment $\sim 1/L$. According to Ref. 3, which extends Pierce's two-dimensional problem to cylindrical geometry, the limiting current density $j_{\lim}$ of a beam, which is uniform in the radial section, through a circular metallic waveguide (bounded at the ends by metallic foils, which form together with the lateral walls an equipotential surface $\Phi = 0$), is

$$j_{\lim} = \frac{m V_0^3}{4\pi e} \left[(2.4/R)^2 + (\pi/L)^2\right]. \tag{1}$$

The maximum increment of the Pierce instability $\delta_{\max}$ and the increment near threshold δ are given by

$$\delta_{max} \cong \frac{V_0}{L} (R/L)^2 ; \quad \delta \cong \frac{V_0}{L} \Delta j, \qquad \Delta j \equiv \frac{j - j_{\lim}}{j_{\lim}} \tag{2}$$

($j = eN_0V_0$, where N_0 and V_0 are the equilibrium density and velocity of electrons in the beam; $e > 0$ is the charge of an electron; and R is the radius of the waveguide). It follows from (2) that $\delta_{\rm PI} \to 0$ for $L \to \infty$. It was assumed that in long ($L \gg R$) systems, in the limit of an infinite waveguide with metallic foil at the inlet, the beam is stable against potential perturbations and, under the assumptions adopted, a current with arbitrary magnitude can be passed through such systems. It is shown below that this is incorrect. A quasineutral beam is likewise unstable in a semiinfinite tube with "perfectly conducting" walls; furthermore, the limiting current is the Pierce current (1), while the increment of the aperiodic potential instability is expressed by Eq. (13).

2. Let us formulate the initial problem. In the unperturbed state, a uniform monoenergetic neutralized electron beam completely fills a semiinfinite waveguide and moves toward its open end along the positive Oz axis. A potential $\Phi = 0$ is maintained on the walls of the waveguide and on the inlet foil ($z = 0$), while an infinite magnetic field, which prevents radial motion of electrons, is applied along the axis of the waveguide. The evolution of the initial perturbation of the electron density (or velocity; this is not significant) is described by the following system of linearized equations and boundary conditions:

$$\left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial z}\right) \delta N + N_0 \frac{\partial \delta V}{\partial z} = 0, \quad \delta N(r, z; t=0) = \delta N^0(r, z) ;$$

$$\left(\frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial z}\right) \delta V = \frac{e}{m} \frac{\partial \delta \Phi}{\partial z}, \quad \delta V(r, z; t=0) = 0 ; \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\,\delta\Phi}{\partial r}\right)+\frac{\partial^2\delta\Phi}{\partial z^2}=4\pi e\,\delta N; \tag{4}$$

$$\begin{aligned}&\delta N(r=R,z;t)=\delta N(r,z=0;t)=0,\\&\delta V(r=R,z;t)=\delta V(r,z=0;t)=0;\end{aligned} \tag{5}$$

$$\delta\Phi(r=R,\ z;t)=\delta\Phi(r,z=0;t)=0,\qquad |\delta\Phi|<\infty. \tag{6}$$

As an additional condition, we require that for $z\to\infty$ the density perturbations $\delta N(z)$ and velocity perturbations $\delta V(z)$ as a function of z be representable as a Fourier integral [see (8)]. We examine only azimuthally symmetrical perturbations, which depend on the cylindrical coordinates (r, z). As far as the dependence on r is concerned, we shall restrict the analysis to solutions that have the form of one of the characteristic modes of the radial part of Poisson's equation (4) with boundary conditions (6). These solutions are Bessel functions $J_0(\lambda_s r)$, where $\lambda_s\equiv\mu_s/R$, and $\mu_s>0$ are roots of the equation $J_0(\mu_s)=0$, $s=1, 2, \ldots$. The restriction to single-mode azimuthally symmetrical solutions is not fundamental and merely simplifies the calculations.

3. Thus we seek the solution of the system (3)–(6) in the form $\delta N(r, z;t)=J_0(\lambda_s r)\times n(z, t)$ and analogously for δV, $\delta\Phi$; furthermore, we assume that $\delta N^0(r, z)=J_0(\lambda_s r)\times n^0(z)$. For such solutions, the initial problem (3)–(6) reduces to an equivalent integral equation for $n_{\sin}$ (k, σ) which represents a Laplace transform with respect to the time variable t and a Fourier sin transform with respect to the coordinate z of the perturbation $n(z, t)$:

$$n_{\sin}(k,\sigma)=\frac{n^0_{\sin}(k)}{2}\left\{\frac{\sigma}{\sigma^2+\omega_-^2(k)}+\frac{\sigma}{\sigma^2+\omega_+^2(k)}\right\}+\frac{n^0_{\cos}(k)}{2}\left\{\frac{\omega_-(k)}{\sigma^2+\omega_-^2(k)}+\right.$$

$$\left.+\frac{\omega_+(k)}{\sigma^2+\omega_+^2(k)}\right\}+\frac{\omega_+(k)-\omega_-(k)}{2\pi}\left\{\frac{\sigma}{\sigma^2+\omega_-^2(k)}-\frac{\sigma}{\sigma^2+\omega_+^2(k)}\right\}\int_0^\infty n_{\sin}(\kappa,\sigma)\frac{\kappa\,d\kappa}{\kappa^2+\lambda_s^2}. \tag{7}$$

Here

$$n_{\sin}(k,\sigma)=\sqrt{\frac{2}{\pi}}\int_0^\infty n(z,\sigma)\sin(kz)\,dz,\quad n(z,\sigma)=\int_0^\infty e^{-\sigma t}n(z,t)dt; \tag{8}$$

$n^0_{\sin,\cos}(k)$ are the sin and cos Fourier transforms of the initial perturbation $n^0(z)$;

$$\omega_\pm(k)=kV_0\left(1\pm\sqrt{\frac{k_c^2+\lambda_s^2}{k^2+\lambda_s^2}}\right) \tag{9}$$

are the fast and slow waves of the space charge of the beam;

$$k_c^2 \equiv \frac{\omega_b^2}{V_0^2} - \lambda_s^2 \equiv \lambda_s^2 \frac{j - j^*}{j^*} \equiv \lambda_s^2 \, \Delta j \, , \tag{10}$$

$$j^* = \frac{m\, V_0^3}{4\pi\, e} \lambda_s^2 \; ; \quad s = 1, 2, \ldots \; ; \tag{11}$$

and, $\omega_b^2 = 4\pi e^2 N_0 / m$. Thus the problem of the evolution of the initial perturbation reduces to the problem of finding the eigenvalues and eigenfunctions for the linear homogeneous integral equation, corresponding to the inhomogeneous equation (7). The right side of this homogeneous equation [the right side of (7) without the terms containing $n^0(k)$] as a function of k, is for known value of σ, an eigenfunction, since the integral on the right side is a constant (C_s). The condition $C_s \neq 0$ gives the dispersion equation, which determines all the natural frequencies σ of our system:

$$1 = \sum_{\nu = +,-} \frac{1}{\pi} \int_0^\infty \frac{\sigma}{\sigma^2 + \omega_\nu^2(k)} \left(\frac{d\omega_\nu(k)}{dk} - \frac{\omega_\nu(k)}{k} \right) dk \tag{12}$$

[the perturbed quantities have the time dependence $\sim \exp(\sigma t)$, where σ are the roots of Eq. (12)].

4. For $k_c^2 > 0$, i.e., for $j > j^*$ Eq. (12) has, first of all, the root $\sigma = 0$, which corresponds to a stationary density wave $n(z, t) \sim \sin(k_c z)$. Secondly, an estimate of integral (12), for $0 < \Delta j \ll 1$, shows that a positive root exists:

$$\sigma \cong 0,2\, \lambda_s\, V_0\, (\Delta j)^{3/2} \equiv 0,2\, \mu_s \frac{V_0}{R} \, (\Delta j)^{3/2}, \quad s = 1, 2, \ldots \, , \tag{13}$$

which gives the linear increment of the aperiodic instability a small distance above the current threshold. This instability is not an extension of the Pierce instability in a bounded tube to the case $L \to \infty$, since it is obtained for other characteristic solutions of the system (3)–(6). We were interested in solutions that are (generally) localized, in regions of finite z, i.e., solutions that can be represented in the form of a wave packet (8) with real $0 < k < \infty$. The given instability, which is insensitive to the boundary $z = L \gg R$, determines the linear stage of formation of a virtual cathode in long systems. The coincidence of the limiting currents (1) and (11) is natural: In both cases, feedback is realized by the same, slow wave of space charge $\omega_-(k)$, which for $k_c^2 > 0$ has, in the laboratory system of coordinates, negative phase velocity for $0 < k < k_c$ and negative group velocity for $0 < k < \sim k_c/2$.

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